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Large Deviations of a Network of Interacting Particles with Sparse Random Connections

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Abstract In this work we determine a process-level Large Deviation Principle (LDP) for a model of interacting particles indexed by a lattice \mathbb{Z}^d . The connections are random, sparse and unscaled, so that the system converges in the large size limit due to the probability of a connection between any two particles decreasing as the system size increases. The particles are also subject to noise (such as independent Brownian Motions). The method of proof is to assume a process-level (or Level 3) LDP for the double-layer empirical measure for the noise and connections, and then apply a series of transformations to this to obtain an LDP for the process-level empirical measure of our system. Although it is not explicitly necessary, we expect that most applications of this work should involve an assumption of stationarity of the probability law for the noise and connections under translations of the lattice, so that the system converges to an ergodic probability law in the large size limit. This work synthesizes the theory of large-size limits of interacting particles with that of random graphs and matrices. It should therefore be relevant to neuroscience and social networks theory in particular.

Keywords Large Deviations, random graph, interacting particles, empirical measure, SDE, lattice

1 Introduction

In this paper we determine a Large Deviation Principle for an asymptotically large system of interacting processes on a lattice with sparse random connections. We are motivated in particular by the study of interacting neurons in neuroscience [9], but this work ought to be adaptable to other phenomena such as mathematical finance, social networks, population genetics or insect swarms [58, 2].

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Classical mean-field models are perhaps the most common method used to scale up from the level of individual particles to the level of populations [3, 67]. For a group of neurons indexed from 1 to N , the evolution equation of a mean field model is typically of the following form (an \mathbb{R}^N -valued SDE)

$$dX_t^j = \left[g(X_t^j) + \frac{1}{N} \sum_{k=1}^N h_t(X_t^j, X_t^k) \right] dt + \sigma(X_t^j) dW_t^j. \quad (1)$$

We set $X_0^j = 0$. Here g is Lipschitz, h is Lipschitz and bounded, and σ is Lipschitz. $(W^j)_{j \in \mathbb{N}}$ are independent Brownian Motions representing internal / external noise. Asymptoting the number of particles N to ∞ , we find that in the limit X^j is independent of X^k (for $j \neq k$), and each X^j is governed by the same law [65]. Since the $(X^j)_{j \in \mathbb{N}}$ become more and more independent, it is meaningful to talk of their mean as being representative of the group as a whole. In reaching this limit, three crucial assumptions have been made: that the external synaptic noise is uncorrelated, that the connections between the particles are homogeneous and that the connections are scaled by the inverse of the size of the system. We will relax each of these assumptions in our model (which is outlined in Section 2.2).

The major difference between the model in Section 2.2 and the mean field model outlined above is the model of the connections. In the mean field model, the network is completely connected, with a uniform connection strength of N^{-1} . In our work, the connections are sparse and random, and typically (although not necessarily) sampled from a spatially-stationary probability law (i.e. a law that is invariant under translations of the lattice). The system converges as $N \rightarrow \infty$ because the probability of a connection between particles decreases as the lattice distance increases. This model bears some similarities to [32], where the connection strengths are also not scaled by N^{-1} , but rather decrease uniformly as the lattice distance increases.

Our network has the structure of a sparse random graph with edges which are directed and weighted. In neuroscience and social network theory in particular, there is an enormous literature devoted to the study of complex networks [64, 58]. There is also a rich mathematical literature on random graphs; see for example [27, 4, 24, 48]. The connection model in this paper may be thought of as an Erdos-Renyi random weighted graph. In other words, as the network grows, the number of vertices increases uniformly, but the connection strength between any two vertices is random, such that the probability of a strong connection decreases as the lattice distance increases. Furthermore the connections are correlated, with the correlation between two connections dependent on the lattice distance between the heads and tails. Broadly speaking, the resulting network structure is that of a ‘small-world’, with a high degree of local connectivity on average, but also some long-range connections [70].

It would be of interest to see if some of the literature on the Large Deviations of random matrices and graphs could improve the results in this paper [1]. For example, [15] find a Large Deviations Principle for Erdos-Renyi random graphs. Although our model is different to theirs, since the probability of an edge existing between two vertices is not uniform throughout the graph,

it seems likely that if their results on the asymptotic clustering of edges could be generalised to our context then some of our bounds could be improved (such as in Lemma 6 where we make the assumption of complete connectivity throughout the cube V_m).

Various authors have analysed the thermodynamic limit of Gibbs systems with disordered interactions, including [71, 60, 16, 45]. There has also been some study recently of the effect of disorder on mean-field coupled oscillators on a lattice, see for instance [18, 42, 50, 49, 13]. In these works, the disorder is added at each lattice point and sampled independently from a fixed distribution. The behaviour of the finite-size system is then studied through its empirical measure. Our work has some formal similarities, except that in our case the disorder is manifested in the random connections, the interactions are sparse and unscaled and we study the behaviour of the process-level empirical measure which captures system-wide correlations. Finally, there has been some recent interest in the asymptotics of the empirical measure of completely connected networks of interacting particles, with the connection strength sampled from a probability distribution and scaled by $N^{-\frac{1}{2}}$ [5, 52, 66, 12, 33, 13].

This article synthesizes large scale interacting particle models (such as mean-field models or Gibbs models) with random graph models, in the manner of (for example) [59]. It has already been demonstrated that non-uniform connections can have a critical effect on the large-scale behaviour of interacting particle models. [63] investigate chaos in neural network models with random weights; in contrast to the model in this paper, they scale the connection strengths by $N^{-\frac{1}{2}}$ so that in the large size limit as $N \rightarrow \infty$, the system converges to a limit thanks to the central limit theorem. [56] study the dynamics of neural networks with complex random topologies, finding conditions under which the network dynamics converges to that of mean-fields in the large size limit. [54] find that networks of Kuromoto oscillators with sparse random connections are more likely to synchronise. [20] find that if the connections are random but balanced, then synchronisation may emerge, but this is strongly dependent on the nature of the connectivity. In addition, it has recently been argued that if there are random connections in systems of interacting particles then this can decrease the variability [69].

The main result of this paper is a Large Deviation Principle (LDP) for the interacting particle model in Section 2.2. A Large Deviation Principle is a very useful mathematical technique that allows us to estimate finite-size deviations of the system from its limit behaviour. There has been much effort in recent years to understand such finite-size phenomena in mathematical models of neural networks - see for instance [8, 11, 68, 30]. More generally, there exists a well-developed literature on the Large Deviations and other asymptotics of weakly-interacting particle systems (see for example [5, 43, 19, 10, 34, 67, 21, 32, 7]). These are systems of N particles, each evolving stochastically, and usually only interacting via the empirical measure. We note also the literature on the long-time behaviour of an infinite set of interacting particles on a lattice, including [17, 47, 44]. Lastly, it should be mentioned that our main result bears some similarities to previous work on

the large deviations of a random walk in a random environment, including [72, 57, 46, 38].

This paper is structured as follows. In Section 2 we outline a general model of interacting particles on a lattice with random connections, and state a large deviation principle under a set of assumptions. In Section 3 we outline an extended example of this theory which satisfies the assumptions of Section 2. This example considers a sparse network of Fitzhugh-Nagumo neurons, with Hebbian learning on the connections, and the probability of a connection between two neurons being given by a Gibbs distribution. The remaining sections are dedicated to the proof of the main result in Theorem 1.

2 Outline of Model and Preliminary Definitions

In this section we outline our finite model of $(2n+1)^d$ stationary interacting particles indexed over the cube V_n . In Subsection 2.3 we outline our assumptions on the model. The main result of this paper is stated in Theorem 1.

2.1 Preliminaries

We must first make some preliminary definitions. If X is some separable topological space, then we denote the σ -algebra generated by the open sets by $\mathcal{B}(X)$, and the set of all probability measures on $(X, \mathcal{B}(X))$ by $\mathcal{P}(X)$. We always endow $\mathcal{P}(X)$ with the topology of weak convergence.

Elements of the processes in this paper are indexed by the lattice points \mathbb{Z}^d : for $j \in \mathbb{Z}^d$ we write $j = (j(1), \dots, j(d))$. Let $V_n \subset \mathbb{Z}^d$ be such that $j \in V_n$ if $|j(m)| \leq n$ for all $1 \leq m \leq d$. The number of elements in V_n is written as $|V_n| := (2n+1)^d$.

For any $s \in [0, T]$, we endow $\mathcal{C}([0, s], \mathbb{R})$ with the norm $\|U\|_s := \sup_{r \in [0, s]} |U_r|$, and we write $\mathcal{T} := \mathcal{C}([0, T], \mathbb{R})$. In the model outlined in the next section, the activity over time of each particle is a \mathcal{T} -valued random variable. The state space for the connections between the particles is taken to be a complete separable metric space \mathfrak{C} , with metric $d_{\mathfrak{C}}(\cdot, \cdot)$. Let $\pi^{V_m} : \mathcal{T}^{\mathbb{Z}^d} \rightarrow \mathcal{T}^{V_m}$ be the projection $\pi^{V_m}(X) := (X^j)_{j \in V_m}$. We endow $\mathcal{T}^{\mathbb{Z}^d}$ with the cylindrical topology (generated by sets $O \subset \mathcal{T}^{\mathbb{Z}^d}$ such that $\pi^{V_m}O$ is open in \mathcal{T}^{V_m}). Let $d_m^{\mathcal{P}}$ be the Levy-Prokhorov metric on $\mathcal{P}(\mathcal{T}^{V_m})$ generated by the norm $\|x\|_{T,m} := \sum_{j \in V_m} \|x^j\|_T$ on \mathcal{T}^{V_m} . The following metric on $\mathcal{P}(\mathcal{T}^{\mathbb{Z}^d})$ metrizes weak convergence,

$$d^{\mathcal{P}}(\mu, \nu) := \sum_{j=1}^{\infty} \min \left(2^{-j}, d_j^{\mathcal{P}}(\pi_{V_j}^{\mathcal{P}} \mu, \pi_{V_j}^{\mathcal{P}} \nu) \right), \quad (2)$$

where $\pi_{V_j}^{\mathcal{P}}$ is the projection onto the marginal in $\mathcal{P}(\mathcal{T}^{V_j})$. Thanks to the Komolgorov Extension Theorem, $d^{\mathcal{P}}$ is complete (i.e. each Cauchy Sequence converges to a unique limit), since each $d_j^{\mathcal{P}}$ is complete over $\mathcal{P}(\mathcal{T}^{V_j})$ [6].

It may be noted that many papers on the convergence of networks of interacting particles use the Wasserstein Metric rather than the above Levy Prokhorov metric. However the implementation of the Wasserstein metric is complicated by the fact that the map Ψ^m (which maps the noise and connections to the solution and is defined in (41)) is not continuous with respect to the cylindrical topology. This is why we use the Levy-Prokhorov metric in this paper.

2.2 Outline of Model and Main Result

For $n \in \mathbb{Z}^+$, there are $|V_n|$ particles in our network. There are three components to the dynamics of our network: the internal dynamics term \mathbf{b}_s , the interaction term $\Lambda_s^k(J^{n,j,k}, U^j, U^{(j+k) \bmod V_n})$ and the noise term $W_t^{n,j}$. The form of our interaction term differs from standard mean-field models in that it is not scaled by some function of $|V_n|$, and it is not homogeneous throughout the network. It depends also on the random connection $J^{n,j,m}$ between the particles at nodes j and $(j+m) \bmod V_n$ (here $m \in V_n$ is the vector distance between the two particles relative to the toroidal topology). $(J^{n,j,m})_{j,m \in V_n}$ are correlated \mathfrak{C} -valued random variables. Further on, in Assumption 5, we will assume that there exists a null-connection \mathfrak{N} such that $\Lambda_s^k(\mathfrak{N}, \cdot, \cdot)$ is zero, and that as the network size asymptotes to infinity most connections are of this form (otherwise the system would blow-up). A simple example of a possible model is that of an unweighted sparse random graph, where $J^{n,j,m} \in \{0, 1\}$ (1 corresponding to there being a connection and $0 := \mathfrak{N}$ to no connection); we outline a Gibbsian model of this form in the following section, which could in principle be fitted to experimental data. Notice also that the function itself can depend on the lattice distance k between the particles (the distance being taken modulo V_n), so that for example delays in the transmission and learning may be taken into account (such as in [32]).

We assume that $W^n := (W_t^{n,j})_{j \in V_n, t \in [0, T]}$ is a \mathcal{T}^{V_n} -valued random variable (such as independent Brownian motions). The system we study in this paper is governed by the following evolution equation: for $j \in V_n$ and $t \in [0, T]$,

$$U_t^j = \int_0^t \left(\mathbf{b}_s(U^j) + \sum_{k \in V_n} \Lambda_s^k(J^{n,j,k}, U^j, U^{(j+k) \bmod V_n}) \right) ds + W_t^{n,j}. \quad (3)$$

Here $(j+k) \bmod V_n := l \in V_n$, such that $(j(p)+k(p)) \bmod (2n+1) = l(p)$ for all $1 \leq p \leq d$. Thus one may think of the particles as existing on a d -dimensional torus. It is noted in Lemma 1 that there exists a unique solution to (3) for every $W^n \in \mathcal{T}^{V_n}$. The thrust of this article is to understand the asymptotic behaviour of the network as $n \rightarrow \infty$.

We now define the process-level empirical measure $\hat{\mu}^n$. This is a reduced macroscopic variable which is used to study global phenomena, such as phase transitions, chaos or synchrony [28]. Let $S^k : \mathcal{T}^{\mathbb{Z}^d} \rightarrow \mathcal{T}^{\mathbb{Z}^d}$ (for some $k \in \mathbb{Z}^d$)

be the shift operator (i.e. $(S^k x)^m := x^{m+k}$). Let $\mathcal{P}_S(\mathcal{T}^{\mathbb{Z}^d})$ be the set of all stationary probability measures, i.e. such that for all $k \in \mathbb{Z}^d$,

$$\mu \circ (S^k)^{-1} = \mu.$$

Denote the empirical measure $\hat{\mu}^n : \mathcal{T}^{V_n} \rightarrow \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}^d})$ by

$$\hat{\mu}^n(X) := \frac{1}{|V_n|} \sum_{j \in V_n} \delta_{S^j \tilde{X}}, \quad (4)$$

where $\tilde{X} \in \mathcal{T}^{\mathbb{Z}^d}$ is the V_n -periodic interpolant, i.e. $\tilde{X}^j := \tilde{X}^{j \bmod V_n}$. If $X \in \mathcal{T}^{\mathbb{Z}^d}$, then in a slight abuse of notation we write $\hat{\mu}^n(X) := \hat{\mu}^n(\pi^{V_n} X)$.

We now outline the main result of this paper.

Theorem 1 *Let the law of $\hat{\mu}^n(U)$ be $\Pi^n \in \mathcal{P}(\mathcal{P}_S(\mathcal{T}^{\mathbb{Z}^d}))$. Under the assumptions outlined in Section 2.3, $(\Pi^n)_{n \in \mathbb{Z}^+}$ satisfy a Large Deviation Principle with good rate function I (i.e. I has compact level sets). That is, for all closed subsets A of $\mathcal{P}_S(\mathcal{T}^{\mathbb{Z}^d})$,*

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{|V_n|} \log \Pi^n(A) \leq - \inf_{\gamma \in A} I(\gamma). \quad (5)$$

For all open subsets O of $\mathcal{P}_S(\mathcal{T}^{\mathbb{Z}^d})$,

$$\underline{\lim}_{n \rightarrow \infty} \frac{1}{|V_n|} \log \Pi^n(O) \geq - \inf_{\gamma \in O} I(\gamma). \quad (6)$$

The rate function is

$$I(\mu) := \inf_{\nu \in \mathcal{P}_{\bar{S}}(\bar{\mathcal{T}}^{\mathbb{Z}^d})} \left\{ I_Y(\nu) : \dot{\Psi}(\nu) = \mu \right\},$$

where $\mathcal{P}_{\bar{S}}(\bar{\mathcal{T}}^{\mathbb{Z}^d})$ is defined in (10), $\dot{\Psi}$ is defined in (40),(43),(44) and I_Y is defined in Assumption 4.

From now on, if a sequence of probability laws satisfies (5) and (6) for some I with compact level sets, then to economise space we say that it satisfies an LDP with a good rate function I . Theorem 1 is useful because it allows us to understand the asymptotic behaviour of averages over the entire network, as is noted in the following corollary.

Corollary 1 *Suppose that there is a unique $\mu_Y \in \mathcal{P}_{\bar{S}}(\bar{\mathcal{T}}^{\mathbb{Z}^d})$ such that $I_Y(\mu_Y) = 0$ (see Assumption 4 for the definitions of these terms). For some positive integer q , let $g : \mathcal{T}^{V_q} \rightarrow \mathbb{R}$ be continuous and bounded. Then, writing $H_n = \frac{1}{|V_n|} \sum_{j \in V_n} g(S^j \cdot \tilde{U})$,*

$$\lim_{n \rightarrow \infty} H_n = \mathbb{E}^{\dot{\Psi}(\mu_Y)}[g],$$

and the sequence of laws $(\Pi_g^n)_{n \in \mathbb{Z}^+} \subset \mathcal{P}(\mathbb{R})$ of H_n satisfies an LDP with good rate function.

2.3 Assumptions

We employ the following assumptions.

In many interacting particle models, such as the Fitzhugh-Nagumo model in Section 3, the internal dynamics term \mathfrak{b}_s is not Lipschitz. In particular, \mathfrak{b}_s is usually strongly decaying when the activity is greatly elevated, so that \mathfrak{b}_s always acts to restore the particle to its resting state. This decay is necessary in order for the neurons to exhibit their characteristic ‘spiking’ behaviour. The following assumptions can accommodate this non-Lipschitz behaviour.

Assumption 1 *Assume that \mathfrak{b}_t is continuous on $[0, T] \times \mathcal{T}$, that for fixed t \mathfrak{b}_t is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathcal{C}([0, t], \mathbb{R}))$ measurable and that for each positive constant A ,*

$$\sup_{t \in [0, T], \{X \in \mathcal{T} : \|X\|_T \leq A\}} |\mathfrak{b}_t(X)| < \infty. \quad (7)$$

We assume that there exists a positive constant C such that if $Z_t^j \geq 0$, then

$$\mathfrak{b}_t(Z^j) \leq C \|Z^j\|_t$$

and if $Z_t^j \leq 0$, then

$$\mathfrak{b}_t(Z^j) \geq -C \|Z^j\|_t.$$

If $X_t^j \geq Z_t^j$, then

$$\mathfrak{b}_t(X^j) - \mathfrak{b}_t(Z^j) \leq C \|X^j - Z^j\|_t,$$

and if $X_t^j \leq Z_t^j$, then

$$\mathfrak{b}_t(X^j) - \mathfrak{b}_t(Z^j) \geq -C \|X^j - Z^j\|_t.$$

This paper is primarily intended to model sparse networks. The system converges, not because there are $O(|V_n|)$ connections which decrease in magnitude either uniformly as the network grows (as in mean-field models) or as the lattice distance increases (as in [32]), but rather because the probability of there being a connection between two particles decreases as the lattice distance increases. We are primarily concerned with models where, if there is a connection between two particles, then it is strong, even if they are a long way apart on the lattice.

We assume that there exists $\mathfrak{N} \in \mathfrak{C}$ (this is a ‘null-connection’) such that the interaction is zero, that is for all $k \in \mathbb{Z}^d$ and $s \in [0, T]$, $\Lambda_s^k(\mathfrak{N}, \cdot, \cdot) = 0$. For $J \in \mathfrak{C}$, write

$$|J|_{\mathfrak{C}} := d_{\mathfrak{C}}(\mathfrak{N}, J). \quad (8)$$

The interactions $\Lambda_s^k(\cdot, \cdot, \cdot)$ in many of the networks that we want to study are typically nonlinear. See for example the model of the interactions in [3], and also the example model in Section 3.

Assumption 2 We assume that for some positive constant $C_J \in \mathbb{R}^+$, for all $x \in \mathfrak{C}$,

$$|x|_{\mathfrak{C}} \leq C_J.$$

Assume that for all $k \in \mathbb{Z}^d$, $\Lambda_s^k(\cdot, \cdot, \cdot)$ is continuous on $[0, T] \times \mathfrak{C} \times \mathcal{T} \times \mathcal{T}$, and for each $s \in [0, T]$, Λ_s^k is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathfrak{C}) \times \mathcal{B}(\mathcal{C}([0, s], \mathbb{R})) \times \mathcal{B}(\mathcal{C}([0, s], \mathbb{R}))$ -measurable. For all $x \in \mathfrak{C}$, $t \in [0, T]$, $U, X, Z \in \mathcal{T}$ and $k \in \mathbb{Z}^d$

$$\begin{aligned} |\Lambda_t^k(x, U, X) - \Lambda_t^k(x, U, Z)| &\leq |x|_{\mathfrak{C}} \|X - Z\|_t, \\ |\Lambda_t^k(x, U, X) - \Lambda_t^k(x, Z, X)| &\leq |x|_{\mathfrak{C}} \|U - Z\|_t, \\ |\Lambda_t^k(x, U, X) - \Lambda_t^k(y, U, X)| &\leq (\|U\|_t + \|X\|_t) d_{\mathfrak{C}}(x, y). \end{aligned}$$

We also assume that

$$|\Lambda_t^k(x, X, Z)| \leq |x|_{\mathfrak{C}} (1 + \|X\|_t). \quad (9)$$

Since, in contrast to the great majority of papers on interacting particles surveyed in the introduction, the magnitude of the interactions is independent of the network size, the effect of the interactions grows rapidly as the entire system grows. The following bound ensures that the system does not blowup as the size scales to infinity.

Assumption 3 We assume that there exist positive constants $\mathfrak{a}_1, \mathfrak{a}_2 > 0$, a constant $\rho > 1$ and a positive integer \mathfrak{m}_0 such that for all $m \geq \mathfrak{m}_0$,

$$\begin{aligned} &\mathbb{E} \left[\exp \left(\mathfrak{a}_1 |V_m|^{2\rho+2} \exp((4+2\rho)TC_J|V_m|) \sum_{j \in V_n} \left(\sum_{k \notin V_m} |J^{n,j,k}|_{\mathfrak{C}} \right)^2 \right) \right] \\ &\leq \exp \left(\mathfrak{a}_2 |V_n| \right) \\ &\mathbb{E} \left[\exp \left(\mathfrak{a}_1 |V_m|^{\rho+1} \exp((3+\rho)TC_J|V_m|) \sum_{j \in V_n} \sum_{k \notin V_m} |J^{n,j,k}|_{\mathfrak{C}} \right) \right] \\ &\leq \exp \left(\mathfrak{a}_2 |V_n| \right) \end{aligned}$$

Assumptions 2 and 3 are strongly interrelated: for example one could make Assumption 3 less restrictive by making Assumption 2 more restrictive.

The heart of the method in this paper is to recognize that the solution U of (3) can be written as an ergodic transformation in an ambient higher-dimensional space $\bar{\mathcal{T}}^{\mathbb{Z}^d}$ containing both the noise and the connections. The Large Deviation Principle for U can then be obtained from the (assumed) Large Deviation Principle of W^n and J^n using transformation methods, since the empirical measure for U can be written as a function of the *double layer* empirical measure which incorporates both W^n and J^n (see Lemma 1). While it is not explicitly necessary, in most applications of this theory we expect that the probability laws of W^n and J^n are invariant under translations of the lattice / rotations of the torus (i.e. they are stationary). This is a common assumption when one wants to prove an LDP for the process-level empirical measure (see the papers surveyed in the introduction).

We now define this ambient space $\bar{\mathcal{T}}^{\mathbb{Z}^d}$. Let $\bar{\mathcal{T}} = \mathcal{T} \times \mathfrak{C}^{\mathbb{Z}^d}$, with $\mathfrak{C}^{\mathbb{Z}^d}$ endowed with the cylindrical topology and $\bar{\mathcal{T}}$ with the product topology. Note that $\bar{\mathcal{T}}$ is separable. For $l \in \mathbb{Z}^d$, let $\bar{S}^l : \bar{\mathcal{T}}^{\mathbb{Z}^d} \rightarrow \bar{\mathcal{T}}^{\mathbb{Z}^d}$ be $(\bar{S}^l \cdot \mathcal{X})^j := \mathcal{X}^{j+l}$. In our toroidal topology, \bar{S} corresponds to uniformly rotating both the particles and the connections; a similar construction has been used for the Edwards-Anderson spin glass model in [60, Section 2.8]. Let $\mathcal{P}_{\bar{S}}(\bar{\mathcal{T}}^{\mathbb{Z}^d})$ be the set of all stationary measures, i.e. such that $\mu \in \mathcal{P}_{\bar{S}}(\bar{\mathcal{T}}^{\mathbb{Z}^d})$ if and only if for all $k \in \mathbb{Z}^d$,

$$\mu \circ (\bar{S}^k)^{-1} = \mu. \quad (10)$$

For $j \in V_n$ and $k \notin V_n$, let $J^{n,j,k} = \mathfrak{N}$ (note that this doesn't affect the dynamics in (3)). Let $\tilde{\mathcal{Y}}^n$ be the $\bar{\mathcal{T}}^{\mathbb{Z}^d}$ -valued random variable such that $\tilde{\mathcal{Y}}^{n,p} := (W^{n,p \bmod V_n}, \{J^{n,p \bmod V_n,k}\}_{k \in \mathbb{Z}^d})$. Let the law of

$$\hat{\mu}^n(\mathcal{Y}^n) := \frac{1}{|V_n|} \sum_{k \in V_n} \delta_{\bar{S}^k \tilde{\mathcal{Y}}^n} \quad (11)$$

be $\Pi_{\mathcal{Y}}^n \in \mathcal{P}(\mathcal{P}_{\bar{S}}(\bar{\mathcal{T}}^{\mathbb{Z}^d}))$. $\hat{\mu}^n(\mathcal{Y}^n)$ is sometimes known as the *double-layer* empirical measure, as it incorporates both the noise and the random environment [18].

Assumption 4 *The series of laws $(\Pi_{\mathcal{Y}}^n)_{n \in \mathbb{Z}^+}$ is assumed to satisfy a Large Deviation Principle with good rate function $I_{\mathcal{Y}} : \mathcal{P}_{\bar{S}}(\bar{\mathcal{T}}^{\mathbb{Z}^d}) \rightarrow \mathbb{R}^+$.*

The above assumption is not as restrictive as it may appear. In many proofs of level-3 Large Deviations results, one starts from an i.i.d process, and applies standard transformation methods (either an exponential change of measure, as with LDPs for Gibbs distributions [39], or a moving average transformation [25, 31, 32]). If W^n is independent of J^n , and the empirical measures of $\hat{\mu}^n(W^n)$ and $\hat{\mu}^n(J^n)$ each satisfy LDPs obtained through these transformations, then generally $\hat{\mu}^n(\mathcal{Y}^n)$ will satisfy an LDP as well. An example of this is given in Section 3.

Assumption 5 *It is assumed that there exist positive constants $\mathbf{c}_1, \mathbf{c}_2$ such that*

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{|V_n|} \log \mathbb{E} \left[\exp \left(\mathbf{c}_1 \sum_{j \in V_n} \|W^{n,j}\|_T^2 \right) \right] \leq \exp(|V_n| \mathbf{c}_2). \quad (12)$$

3 An Example Application: A Fitzhugh-Nagumo Neural Network with Gibbsian Random Connectivity and Learning

In this section we outline an example from neuroscience of a model satisfying (3) and the assumptions of Section 2.3. Most of these assumptions are satisfied relatively easily, and we therefore omit the proofs, except for Assumption 4, for which we provide a proof in Theorem 2. Of course there are many other sorts of model which would fit the framework of this paper.

We take the internal dynamics to be that of the Fitzhugh-Nagumo model, the connection matrix to be random and Gibbsian, with the interaction terms

driven by the firing rates of the pre and post synaptic neurons, and the connections evolving according to a learning rule. We take $d \in \{1, 2, 3\}$. For $j \in V_n$, the governing equations are

$$dv_t^j = dW_t^{n,j} + \left(\sum_{k \in V_n} G_t^k(J^{n,j,k}, v^j, v^{(j+k) \bmod V_n}) \times \right. \quad (13)$$

$$\left. f(v_t^j) f(v_t^{(j+k) \bmod V_n}) + v_t^j - \frac{1}{3}(v_t^j)^3 - w_t^j \right) dt,$$

$$dw_t^j = (v_t^j + \mathbf{a} - \mathbf{c}w_t^j) dt. \quad (14)$$

We take $w_0^j = v_0^j = 0$ as initial conditions. Here \mathbf{a} and \mathbf{c} are positive constants, and f is bounded and Lipschitz. The internal dynamics of the above equation is that of the famous Fitzhugh-Nagumo model [35, 53, 36, 37]. This model distills the essential mathematical features of the Hodgkin-Huxley model, yielding excitation and transmission properties from the analysis of the biophysics of sodium and potassium flows. The variable v is the ‘fast’ variable which corresponds approximately to the voltage, and w is the ‘slow’ recovery variable which is dominant after the generation of an action potential.

We may reduce this to a one-dimensional equation by noticing that the solution of (14) is

$$w_t^j = \mathbf{c}^{-1} \int_0^t \exp(-\mathbf{c}(t-s)) (v_s^j + \mathbf{a}) ds.$$

Hence we identify $U_t^j := v_t^j$ and

$$\mathbf{b}_t(U^j) := U_t^j - \frac{1}{3}(U_t^j)^3 + \mathbf{c}^{-1} \int_0^t \exp(-\mathbf{c}(t-s)) (U_s^j + \mathbf{a}) ds. \quad (15)$$

For simplicity, we take the noise $\{W^{n,j}\}_{j \in V_n}$ to be independent Brownian motions. It has been proved in [32, Lemma 17] that Assumption 5 is satisfied.

3.1 Model of Interactions

The interaction term is a simplification of the chemical synapse models in [23, 29]. It is assumed that the existence of a connection between neurons j and k is random and determined by the random variable $J^{n,j,k} \in \{0, 1\}$. The model of $J^{n,j,k}$ is outlined in the following section. In the formalism of the previous section, $\mathfrak{C} = \{0, 1\}$, with $\mathfrak{N} = 0$. Letting \bar{J} be the maximal connection strength (so that $C_J = \bar{J}$ in Assumption 2), we define $d_{\mathfrak{C}}(0, 1) = \bar{J}$ and $d_{\mathfrak{C}}(0, 0) = d_{\mathfrak{C}}(1, 1) = 0$. We specify that

$$G_t^k(0, \cdot, \cdot) = 0.$$

Otherwise, we take $G_s^k(1, \cdot, \cdot)$ to evolve according to the following learning rule. We use the following classical Hebbian Learning model (refer to [41] for a more detailed description, and in particular equation 10.6) to specify G_t^k .

The ‘activity’ of neuron j at time t is given as $\mathfrak{v}(U_t^j)$. Here $\mathfrak{v} : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous, positive and bounded. The evolution equation is defined to be, for $X, Y \in \mathcal{T}$,

$$\frac{d}{dt}G_t^k(1, X, Y) = J^{corr} (\bar{J} - G_t^k(1, X, Y)) \mathfrak{v}(X_t) \mathfrak{v}(Y_t) - J^{dec} G_t^k(1, X, Y). \quad (16)$$

Here $J^{corr}, J^{dec}, \bar{J}$ are non-negative constants (if we let them be zero then we obtain weights which are constant in time). Initially, if $J^{n,j,k} \neq \mathfrak{N}$, we stipulate that

$$G_0^k(1, X, Y) := G_{ini} \quad (17)$$

where $G_{ini} \in [0, \bar{J}]$ is a constant stipulating the initial strength of the weights. It is straightforward to show that there is a unique solution to the above differential equation for all $X, Y \in \mathcal{C}([0, T], \mathbb{R}^2)$. One may show that $G_t^k(1, \cdot, \cdot) \leq \bar{J}$. In effect, the solution defines $G_t^k(1, \cdot, \cdot)$ as a function $\mathcal{C}([0, t], \mathbb{R}) \times \mathcal{C}([0, t], \mathbb{R}) \rightarrow \mathbb{R}$, which can be shown to be uniformly Lipschitz in both of its variables, where $\mathcal{C}([0, t], \mathbb{R})$ is endowed with the supremum norm.

3.2 Gibbsian Model of the Random Connections

We use Gibbs Measures to specify the correlations between the random connections (see [39, 55] for a more detailed introduction). In neuroscience, Gibbs measures have already been extensively employed to study spiking neurons [14], and it seems reasonable that they could also be used to study neuronal network models. One such interesting application is given in [62]. It seems reasonable to think that one could motivate the particular choice of the potential functions Φ_A (which are outlined further below) through experimental data on the connectivity of the brain [64, 59, 62]. Gibbs measures are also used extensively in social network theory, where they are often known as *exponential weighted graphs* [58, 51].

The main result of this section is Theorem 2. This theorem is a modification of a standard result for Gibb’s Measures, and it satisfies Assumption 4. To define the Gibbs Measures, we require some more notation.

Let \mathcal{L} be the set of all finite subsets of \mathbb{Z}^{2d} and \mathcal{L}_* the set of all finite subsets containing at least one point of the form $(0, k)$ for any $k \in \mathbb{Z}^d$. Let $\bar{\mathcal{T}}^B = \{(Y^j, \omega^{j,k})_{(j,k) \in B} | Y^j \in \mathcal{T} \text{ and } \omega^{j,k} \in \mathfrak{C}\}$. For any $\mu \in \mathcal{P}(\bar{\mathcal{T}}^{\mathbb{Z}^d})$ and $B \in \mathcal{L}$, let the marginal of μ over the indices in B be $\mu^B \in \mathcal{P}(\bar{\mathcal{T}}^B)$. Let $\tilde{S}^j : \mathfrak{C}^{\mathbb{Z}^{2d}} \rightarrow \mathfrak{C}^{\mathbb{Z}^{2d}}$ be $(\tilde{S}^j \omega)^{k,l} = \omega^{j+k,l}$ and $\bar{V}_q \subset \mathbb{Z}^{2d}$ be

$$\bar{V}_q = \{(j, k) \in \mathbb{Z}^{2d} | j, k \in V_q\}. \quad (18)$$

As noted above, $J^{n,j,k}$ is a \mathfrak{C} -valued random variable, where $\mathfrak{C} = \{0, 1\}$, and $\mathfrak{N} = 0$. We define $d_{\mathfrak{C}}(0, 1) = d_{\mathfrak{C}}(1, 0) = \bar{J}$ and $d_{\mathfrak{C}}(0, 0) = d_{\mathfrak{C}}(1, 1) = 0$. We specify that the law $\mathcal{Q}^n \in \mathcal{P}(\bar{\mathcal{T}}^{V_n})$ of (W^n, J^n) is $\mathcal{Q}_n^{\bar{V}_n}$; i.e. it is the marginal over \bar{V}_n of \mathcal{Q}_n , which is defined further below.

The correlations in the distribution of J^n are specified through the *potential* functions $\Phi_A : \mathfrak{C}^{\mathbb{Z}^{2d}} \rightarrow \mathbb{R}$, for $A \in \mathcal{L}_*$. We require that $\Phi_A(\cdot)$ is $\mathcal{B}(\mathfrak{C}^A)$ -measureable, and the following assumption.

Assumption 6 *We assume that for all $p \in \mathbb{Z}^+$,*

$$\sum_{j \in \mathbb{Z}^d} \sum_{A \in \mathcal{L}_*, \tilde{S}^j A \cap \bar{V}_p \neq \emptyset} \sup_{\omega \in \mathfrak{C}^{\mathbb{Z}^{2d}}} |\Phi_{\tilde{S}^j A}(\omega)| < \infty, \quad (19)$$

It is a consequence of the above assumption that

$$\lim_{p \rightarrow \infty} \sum_{A \in \mathcal{L}_*, A \not\subset \bar{V}_p} \sup_{\zeta \in \mathfrak{C}^{\mathbb{Z}^{2d}}} |\Phi_A(\zeta)| = 0. \quad (20)$$

We now define $\mu_0 \in \mathcal{P}(\bar{\mathcal{T}}^{\mathbb{Z}^d})$, the base probability measure against which the Gibbsian correlations are defined. We define it such that the bounds on the number of null connections in Assumption 3 are satisfied. Suppose that μ_0 is the probability law of the random variables $(W^m, \omega^{m,k})_{m,k \in \mathbb{Z}^d}$. We assume that, under μ_0 , the variables $\{W^m, \omega^{m,k}\}_{m,k \in \mathbb{Z}^d}$ are mutually independent. The variables $\{W^m\}_{m \in \mathbb{Z}^d}$ are independent Brownian motions, and for some integer \mathfrak{m}_0 , for all $m \geq \mathfrak{m}_0$,

$$\begin{aligned} \mu_0(\omega^{j,m} \neq \mathfrak{N}) &= \exp(-v \exp(|V_m|^\gamma)) \\ \mu_0(\omega^{j,m} = \mathfrak{N}) &= 1 - \exp(-v \exp(|V_m|^\gamma)), \end{aligned}$$

for constants $v > 0$ and $\gamma > 1$. It does not matter how we define $\mu_0(\omega^{j,m} \neq \mathfrak{N})$ for $m < \mathfrak{m}_0$.

We define $\mathcal{Q}_m \in \mathcal{P}_{\bar{S}}(\bar{\mathcal{T}}^{\mathbb{Z}^d})$ to be a stationary Gibbs Measure with the conditional probability measures, for $B \in \mathcal{L}_*$, $\mathcal{Q}_m^B(\mathcal{Z}|\mathcal{X}) \in \mathcal{P}(\bar{\mathcal{T}}^B)$. This is defined to be, (writing, for $\mathcal{Z} \in \bar{\mathcal{T}}^B$ and $\mathcal{Z} := (Y^j, \omega^{j,k})$ for $(j,k) \in B$ and $\mathcal{X} \in \bar{\mathcal{T}}^{\mathbb{Z}^{2d}-B}$, where $\mathcal{X} = (X^j, \eta^{j,k})$, for $(j,k) \in \mathbb{Z}^{2d} - B$),

$$\frac{d\mathcal{Q}_m^B}{d\mu_0^B}(\mathcal{Z}|\mathcal{X}) = (Z_m^B(\eta))^{-1} \exp\left(-\sum_{j \in \mathbb{Z}^d} \sum_{A \in \mathcal{L}_*, \tilde{S}^j A \cap B \neq \emptyset} \Phi_{\tilde{S}^j A}(\zeta(m))\right), \quad (21)$$

$$\text{where } \zeta(m)^{j,k} = \mathfrak{N} \text{ if } k \notin V_m, \text{ otherwise} \quad (22)$$

$$\zeta(m)^{j,k} = \omega^{j,k} \text{ if } (j,k) \in B, \text{ otherwise} \quad (23)$$

$$\zeta(m)^{j,k} = \eta^{j,k} \text{ and} \quad (24)$$

$Z_m^B(\eta)$ is the appropriate normalisation constant. It may be observed that \mathcal{Q}_m is a Gibbs measure. It is straightforward to show that Assumption 3 is satisfied.

For any Polish space \mathfrak{Z} , and $\mu, \nu \in \mathcal{P}(\mathfrak{Z})$, let the relative entropy be written as

$$R(\mu||\nu) = \mathbb{E}^\mu \left[\log \frac{d\mu}{d\nu} \right], \quad (25)$$

in the case that $\mu \ll \nu$, else otherwise $R(\mu||\nu) = \infty$. Let $\mathfrak{X}_m^n = \{(j, k) \in \mathbb{Z}^{2d} : j \in V_n \text{ and } k \in V_m\}$ and $\mathfrak{X}_\infty^n = \{(j, k) \in \mathbb{Z}^{2d} : j \in V_n\}$. For any $\mu \in \mathcal{P}_{\bar{\mathcal{S}}}(\bar{\mathcal{T}}^{\mathbb{Z}^d})$, define the *specific relative entropy* with respect to μ_0 to be

$$h(\mu||\mu_0) = \lim_{n \rightarrow \infty} \frac{1}{|V_n|} R(\mu^{\mathfrak{X}_\infty^n} || \mu_0^{\mathfrak{X}_\infty^n}). \quad (26)$$

Define $\Gamma_m : \mathcal{P}_{\bar{\mathcal{S}}}(\bar{\mathcal{T}}^{\mathbb{Z}^d}) \rightarrow \mathbb{R}$,

$$\Gamma_m(\mu) = \mathbb{E}^\mu \left[\sum_{A \in \mathcal{L}_*} \frac{1}{|A|_*} \Phi_A(\zeta(\omega)) \right] + \lim_{n \rightarrow \infty} \frac{1}{|V_n|} \log Z_m^{\mathfrak{X}_m^n}, \quad (27)$$

where $\zeta(\omega)^{j,k} = \mathfrak{N}$ if $k \notin V_m$, else otherwise $\zeta(\omega)^{j,k} = \omega^{j,k}$, and

$$|A|_* := |\{j \in \mathbb{Z}^d | (j, k) \in A \text{ for some } k \in \mathbb{Z}^d\}|.$$

The following limit exists, uniformly in μ , as a consequence of Assumption 6,

$$\Gamma(\mu) = \lim_{m \rightarrow \infty} \Gamma_m(\mu). \quad (28)$$

Let $\Pi_{\mathcal{Y}}^n \in \mathcal{P}(\mathcal{P}_{\bar{\mathcal{S}}}(\bar{\mathcal{T}}^{\mathbb{Z}^d}))$ be the law of $\hat{\mu}(\mathcal{Y}^n)$ under \mathcal{Q}_n (recall the definition of the double layer empirical measure $\hat{\mu}^n(\mathcal{Y}^n)$ in (11)).

Theorem 2 *The sequence of laws $(\Pi_{\mathcal{Y}}^n)_{n \in \mathbb{Z}^+}$ satisfies a Large Deviation Principle with good rate function $I_{\mathcal{Y}} : \mathcal{P}_{\bar{\mathcal{S}}}(\bar{\mathcal{T}}^{\mathbb{Z}^d}) \rightarrow \mathbb{R}$, where*

$$I_{\mathcal{Y}}(\mu) = h(\mu||\mu_0) + \Gamma(\mu). \quad (29)$$

Proof Let $\Pi_m^n \in \mathcal{P}(\mathcal{P}_{\bar{\mathcal{S}}}(\bar{\mathcal{T}}^{\mathbb{Z}^d}))$ be the law of $\hat{\mu}^n$ under \mathcal{Q}_m . Now, thanks to [40], the sequence $(\Pi_m^n)_{n \in \mathbb{Z}^+}$ satisfies an LDP with good rate function,

$$I_m(\mu) = h(\mu||\mu_0) + \Gamma_m(\mu). \quad (30)$$

Since I_m converges uniformly to $I_{\mathcal{Y}}$ (since the convergence in (28) is uniform), $I_{\mathcal{Y}}$ must also be a good rate function.

For any measurable $D \subset \mathcal{P}_{\bar{\mathcal{S}}}(\bar{\mathcal{T}}^{\bar{V}_r})$, let $D^n \subset \bar{\mathcal{T}}^{\mathfrak{X}_r^{n+r}}$ be the event

$$D^n = \left\{ \mathcal{Z} \in \bar{\mathcal{T}}^{\mathfrak{X}_r^{n+r}} : \pi_{\bar{V}_r}^{\mathcal{P}} \hat{\mu}^n(\mathcal{Z}) \in D \right\}, \quad (31)$$

where $\pi_{\bar{V}_r}^{\mathcal{P}}$ is the projection onto the indices in \bar{V}_r (as defined in (18)).

We claim that, for any $\delta > 0$, there exists $\mathfrak{p} \in \mathbb{Z}^+$ such that

$$\sup_{p \geq \mathfrak{p}} \overline{\lim}_{n \rightarrow \infty} \left| \frac{1}{|V_n|} \log \mathcal{Q}_p(D^n) - \frac{1}{|V_n|} \log \mathcal{Q}_{\mathfrak{p}}(D^n) \right| \leq \delta, \quad (32)$$

as long as $\overline{\lim}_{n \rightarrow \infty} \frac{1}{|V_n|} \log \mathcal{Q}_{\mathfrak{p}}(D^n) \neq -\infty$. To see this, suppose that $p > \mathfrak{p} > r$ and observe that

$$\begin{aligned} \left| \frac{1}{|V_n|} \log \mathcal{Q}_p(D^n) - \frac{1}{|V_n|} \log \mathcal{Q}_{\mathfrak{p}}(D^n) \right| &= \frac{1}{|V_n|} \left| \log \frac{\mathcal{Q}_p(D^n)}{\mathcal{Q}_{\mathfrak{p}}(D^n)} \right| \\ &\leq \frac{1}{|V_n|} \sup_{\mathcal{Z} \in \mathfrak{C}^{\mathfrak{x}_r^{n+r}}, \mathcal{X} \in \mathfrak{C}^{\mathbb{Z}^{2d} - \mathfrak{x}_r^{n+r}}} \left| \log \frac{d\mathcal{Q}_p^{\mathfrak{x}_r^{n+r}}}{d\mathcal{Q}_{\mathfrak{p}}^{\mathfrak{x}_r^{n+r}}}(\mathcal{Z}|\mathcal{X}) \right|. \end{aligned}$$

Now, using the same definitions as in (21)-(24),

$$\begin{aligned} \frac{d\mathcal{Q}_p^{\mathfrak{x}_r^{n+r}}}{d\mathcal{Q}_{\mathfrak{p}}^{\mathfrak{x}_r^{n+r}}}(\mathcal{Z}|\mathcal{X}) &= \frac{Z_{\mathfrak{p}}^{\mathfrak{x}_r^{n+r}}(\eta)}{Z_p^{\mathfrak{x}_r^{n+r}}(\eta)} \times \\ &\exp \left(\sum_{j \in \mathbb{Z}^d} \sum_{A \in \mathcal{L}_*, \tilde{S}^j A \cap \mathfrak{X}_r^{n+r} \neq \emptyset} [\Phi_{\tilde{S}^j A}(\zeta(\mathfrak{p})) - \Phi_{\tilde{S}^j A}(\zeta(p))] \right). \end{aligned}$$

Now, since $\Phi_{\tilde{S}^j A}(\zeta(\mathfrak{p})) = \Phi_{\tilde{S}^j A}(\zeta(p))$ when $A \subseteq \bar{V}_{\mathfrak{p}}$,

$$\begin{aligned} &\exp \left(\left| \sum_{j \in \mathbb{Z}^d} \sum_{A \in \mathcal{L}_*, \tilde{S}^j A \cap D^n \neq \emptyset} [\Phi_{\tilde{S}^j A}(\zeta(\mathfrak{p})) - \Phi_{\tilde{S}^j A}(\zeta(p))] \right| \right) \\ &\leq \exp \left(\sum_{j \in V_{n+r}} \sum_{A \in \mathcal{L}_*, A \not\subseteq \bar{V}_{\mathfrak{p}}} |\Phi_{\tilde{S}^j A}(\zeta(\mathfrak{p})) - \Phi_{\tilde{S}^j A}(\zeta(p))| \right) \\ &\leq \exp \left(2|V_{n+r}| \sum_{A \in \mathcal{L}_*, A \not\subseteq \bar{V}_{\mathfrak{p}}} \sup_{\zeta \in \mathfrak{C}^{\mathbb{Z}^{2d}}} |\Phi_A(\zeta)| \right) \\ &\leq \exp \left(2|V_{n+r}| \epsilon_{\mathfrak{p}} \right), \end{aligned}$$

where $\epsilon_{\mathfrak{p}} = \sum_{A \in \mathcal{L}_*, A \not\subseteq \bar{V}_{\mathfrak{p}}} \sup_{\zeta \in \mathfrak{C}^{\mathbb{Z}^{2d}}} |\Phi_A(\zeta)|$. This means that $\exp(-2|V_{n+r}| \epsilon_{\mathfrak{p}}) \leq \frac{Z_p^{D_n}}{Z_{\mathfrak{p}}^{D_n}} \leq \exp(2|V_{n+r}| \epsilon_{\mathfrak{p}})$ and therefore

$$\frac{1}{|V_n|} \sup_{\mathcal{Z} \in \mathfrak{C}^{\mathfrak{x}_r^{n+r}}, \mathcal{X} \in \mathfrak{C}^{\mathbb{Z}^{2d} - \mathfrak{x}_r^{n+r}}} \left| \log \frac{d\mathcal{Q}_p^{\mathfrak{x}_r^{n+r}}}{d\mathcal{Q}_{\mathfrak{p}}^{\mathfrak{x}_r^{n+r}}}(\mathcal{Z}|\mathcal{X}) \right| \leq 4 \frac{|V_{n+r}|}{|V_n|} \epsilon_{\mathfrak{p}} \leq 5 \epsilon_{\mathfrak{p}}, \quad (33)$$

for n sufficiently large. Since, thanks to Assumption 6, $\epsilon_{\mathfrak{p}} \rightarrow 0$ as $\mathfrak{p} \rightarrow \infty$, we have established (32).

We are now ready to prove the Large Deviations Bounds. Suppose that $O \subset \mathcal{P}_{\bar{S}}(\bar{\mathcal{T}}^{\mathbb{Z}^d})$ is open and $\nu \in O$. It follows from the cylindrical topology on $\bar{\mathcal{T}}^{\mathbb{Z}^d}$ that there must exist some $r \in \mathbb{Z}^+$ and open set $O_* \subset \mathcal{P}(\bar{\mathcal{T}}^{\bar{V}_r})$ such that $\pi_{\bar{V}_r}^{\mathcal{P}} \nu \in O_*$ and

$$\left\{ \mu \in \mathcal{P}_{\bar{S}}(\bar{\mathcal{T}}^{\mathbb{Z}^d}) : \pi_{\bar{V}_r}^{\mathcal{P}} \mu \in O_* \right\} \subseteq O. \quad (34)$$

It is then a consequence of the LDP for $(\Pi_m^n)_{n \in \mathbb{Z}^+}$ and (32) that for sufficiently large m ,

$$\begin{aligned} \varliminf_{n \rightarrow \infty} \frac{1}{|V_n|} \log \Pi^n(O) &\geq \varliminf_{n \rightarrow \infty} \frac{1}{|V_n|} \log \Pi^n(O_*) \\ &\geq \varliminf_{n \rightarrow \infty} \frac{1}{|V_n|} \log \Pi_m^n(O_*) - \delta \geq -I_m(\nu) - \delta. \end{aligned}$$

Since ν and δ are arbitrary, and $I_m(\nu) \rightarrow I(\nu)$ (thanks to (28)) it must be that

$$\varliminf_{n \rightarrow \infty} \frac{1}{|V_n|} \log \Pi^n(O) \geq -\inf_{\nu \in \mathcal{O}} I(\nu). \quad (35)$$

Conversely, suppose that $A \subseteq \mathcal{P}_S(\bar{\mathcal{T}}^{\mathbb{Z}^d})$ and A is closed. It is a consequence of the cylindrical topology that we may write $A = \bigcap_{r \in \mathbb{Z}^+} A_r$, where $\pi_{V_r}^{\mathcal{P}} A_r$ is closed in $\mathcal{P}(\bar{\mathcal{T}}^{\bar{V}_r})$, and $A_{r+1} \subseteq A_r$. It is then a consequence of the LDP for $(\Pi_m^n)_{n \in \mathbb{Z}^+}$ and (32) that

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \frac{1}{|V_n|} \log \Pi^n(A) &\leq \overline{\lim}_{n \rightarrow \infty} \frac{1}{|V_n|} \log \Pi^n(A_r) \\ &\leq \overline{\lim}_{n \rightarrow \infty} \frac{1}{|V_n|} \log \Pi_m^n(A_r) + \delta \\ &\leq -\inf_{\mu \in A_r} I_m(\mu) + \delta \\ &\leq -\inf_{\mu \in A_r} I_{\mathcal{Y}}(\mu), \end{aligned}$$

upon taking $\delta \rightarrow 0$ and $m \rightarrow \infty$. The LDP now follows from the fact that $I_{\mathcal{Y}}$ is a good rate function, and

$$\lim_{r \rightarrow \infty} \inf_{\mu \in A_r} I_{\mathcal{Y}}(\mu) = \inf_{\mu \in A} I_{\mathcal{Y}}(\mu), \quad (36)$$

as a consequence of [22, Lemma 4.1.6].

4 Proof of Theorem 1

We obtain the LDP for $(\Pi^n)_{n \in \mathbb{Z}^+}$ by applying a series of transformations to the LDP for $(\Pi_{\mathcal{Y}}^n)_{n \in \mathbb{Z}^+}$ (which is assumed in Assumption 4). The proof of our main result in Theorem 1 is outlined just below. The rest of the document consists of lemmas auxiliary to this proof.

We must first define a metric on the space $\mathcal{P}(\bar{\mathcal{T}}^{\mathbb{Z}^d})$, which, as noted in the previous section, is the space in which the double layer empirical measure $\hat{\mu}^n(\mathcal{Y}^n)$ lives. Recalling that $\bar{V}_q := \{(j, k) \in \mathbb{Z}^{2d} | j, k \in V_q\}$ and $\bar{\mathcal{T}}^{\bar{V}_q} := \{(Y^j, \omega^{j,k})_{(j,k) \in \bar{V}_q} | Y^j \in \mathcal{T} \text{ and } \omega^{j,k} \in \mathfrak{C}\}$, let $\bar{d}_q^{\mathcal{P}}$ be the Levy-Prokhorov metric on $\mathcal{P}(\bar{\mathcal{T}}^{\bar{V}_q})$ generated by the following metric on $\bar{\mathcal{T}}^{\bar{V}_q}$,

$$\bar{d}_q(\mathcal{Y}, \mathcal{Z}) := \sum_{j \in V_q} \|R^j - X^j\|_T + \left(\sum_{j,k \in V_q} d_{\mathfrak{C}}(\omega^{j,k}, \beta^{j,k})^2 \right)^{\frac{1}{2}}, \quad (37)$$

where $\mathcal{Y} := (R^j, \omega^{j,k})_{j,k \in V_q}$, $\mathcal{Z} := (X^j, \beta^{j,k})_{j,k \in V_q}$. Let $\pi_{V_q}^{\mathcal{P}} : \mathcal{P}(\bar{\mathcal{T}}^{\mathbb{Z}^d}) \rightarrow \mathcal{P}(\bar{\mathcal{T}}^{V_q})$ be the projection onto the marginal measure over $\bar{\mathcal{T}}^{V_q}$ and define $\bar{d}^{\mathcal{P}}$ be the following complete metric on $\mathcal{P}(\bar{\mathcal{T}}^{\mathbb{Z}^d})$ (which metrizes weak convergence with respect to the cylindrical topology),

$$\bar{d}^{\mathcal{P}}(\mu, \nu) = \sum_{j=1}^{\infty} \min \left(2^{-j}, \bar{d}_j^{\mathcal{P}}(\pi_{V_j}^{\mathcal{P}} \mu, \pi_{V_j}^{\mathcal{P}} \nu) \right). \quad (38)$$

Proof (Proof of Theorem 1)

For $c \in \mathbb{R}^+$, recalling the definition of $\mathcal{P}_{\bar{S}}(\bar{\mathcal{T}}^{\mathbb{Z}^d})$ in (10), let

$$\begin{aligned} \mathcal{A}_c = & \left\{ \mu \in \mathcal{P}_{\bar{S}}(\bar{\mathcal{T}}^{\mathbb{Z}^d}) \mid \mathbb{E}^{\mu} \left[\|X^0\|_T^2 \right] \leq c \text{ and for all } m \geq \mathbf{m}_0 \right. \\ & \mathbb{E}^{\mu} \left[\left(\sum_{k \notin V_m} |\omega^{0,k}|_{\mathfrak{e}} \right)^2 \right] \leq c \exp \left(- (4 + 2\rho) T C_J |V_m| \right) |V_m|^{-2\rho-2} \text{ and} \\ & \left. \mathbb{E}^{\mu} \left[\sum_{k \notin V_m} |\omega^{0,k}|_{\mathfrak{e}} \right] \leq c \exp \left(- (3 + \rho) T C_J |V_m| \right) |V_m|^{-\rho-1} \right\}, \quad (39) \end{aligned}$$

where we recall that \mathbf{m}_0 and ρ are defined in Assumption 3. It may be observed that \mathcal{A}_c is a closed subset of $\mathcal{P}_{\bar{S}}(\bar{\mathcal{T}}^{\mathbb{Z}^d})$.

We now define maps $\Psi, \Psi^m : \bar{\mathcal{T}}^{\mathbb{Z}^d} \rightarrow \mathcal{T}^{\mathbb{Z}^d}$. These are used to transform the LDP for $(\Pi_{\mathcal{Y}}^n)_{n \in \mathbb{Z}^+}$ into an LDP for $(\Pi^n)_{n \in \mathbb{Z}^+}$. Using the above definitions, for $\mathcal{Z} \in \bar{\mathcal{T}}^{\mathbb{Z}^d}$, with $\mathcal{Z}^j := (R^j, (\omega^{j,l})_{l \in \mathbb{Z}^d})$, we define $\Psi(\mathcal{Z}) := X$ to be such that for any $t \in [0, T]$ and $j \in \mathbb{Z}^d$,

$$X_t^j := \int_0^t \left(\mathfrak{b}_s(X^j) + \sum_{k \in \mathbb{Z}^d} \Lambda_s^k(\omega^{j,k}, X^j, X^{j+k}) \right) ds + R_t^j. \quad (40)$$

Define $\Psi^m : \bar{\mathcal{T}}^{\mathbb{Z}^d} \rightarrow \mathcal{T}^{\mathbb{Z}^d}$ to be $\Psi^m(\mathcal{Z}) := Z$, where Z satisfies

$$Z_t^j := \int_0^t \left(\mathfrak{b}_s(Z^j) + \sum_{k \in V_m} \Lambda_s^k(\omega^{j,k}, Z^j, Z^{j+k}) \right) ds + R_t^j. \quad (41)$$

In fact both Ψ and Ψ^m are only well-defined on a subset of $\bar{\mathcal{T}}^{\mathbb{Z}^d}$ (the domain of Ψ^m is explained at the start of Section 5, and the domain of Ψ may be inferred from Lemma 10). Let

$$\tilde{\Psi}^m : \cup_{c \in \mathbb{R}^+} \mathcal{A}_c \rightarrow \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}^d}) \quad (42)$$

be $\tilde{\Psi}^m(\mu) = \mu \circ (\Psi^m)^{-1}$. In Section 5 we prove that $\tilde{\Psi}^m$ is well-defined and continuous on \mathcal{A}_c . It follows from Lemma 1 that $\tilde{\Psi}^n(\hat{\mu}(\mathcal{Y}^n)) = \hat{\mu}^n(U)$.

Now, there exists a continuous function $\tilde{\Psi}^m : \mathcal{P}_{\bar{S}}(\bar{\mathcal{T}}^{\mathbb{Z}^d}) \rightarrow \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}^d})$ such that $\tilde{\Psi}^m(\mu) = \tilde{\Psi}^m(\mu)$ for all $\mu \in \mathcal{A}_m$. To see this, note that we may identify $\mathcal{P}_S(\mathcal{T}^{\mathbb{Z}^d})$ as a closed convex subset of the locally-convex vector space \mathcal{M} of

all finite Borel measures on $\mathcal{T}^{\mathbb{Z}^d}$ (where \mathcal{M} has the topology of weak convergence). Since \mathcal{A}_m is closed, the existence of $\dot{\Psi}^m$ follows from [26, Theorem 4.1].

We define $\dot{\Psi} : \mathcal{P}_{\bar{S}}(\bar{\mathcal{T}}^{\mathbb{Z}^d}) \rightarrow \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}^d})$ as follows.

$$\dot{\Psi}(\mu) = \mu \circ \Psi^{-1} \text{ if } \mu \in \cup_{c \geq 0} \mathcal{A}_c \quad (43)$$

$$\dot{\Psi}(\mu) = \beta \text{ otherwise, for some arbitrary fixed } \beta \in \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}^d}). \quad (44)$$

Note that it does not matter how we define $\dot{\Psi}$ outside of $\cup_{c \geq 0} \mathcal{A}_c$.

We use [22, Theorem 4.2.23] to prove the result. First note that by Assumption 4, the laws $(\Pi_{\mathcal{Y}}^n)_{n \in \mathbb{Z}^+}$ satisfy an LDP with good rate function. To use [22, Theorem 4.2.23], since $\tilde{\Psi}^n(\hat{\mu}(\mathcal{Y}^n)) = \hat{\mu}^n(U)$ (as noted in Lemma 1), we must first verify the ‘exponentially good approximations’ property, namely that for arbitrary $\alpha, \delta > 0$, there exists an m such that

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{|V_n|} \log \mathbb{P} \left(d^{\mathcal{P}}(\dot{\Psi}^m(\hat{\mu}^n(\mathcal{Y}^n)), \tilde{\Psi}^n(\hat{\mu}^n(\mathcal{Y}^n))) > \delta \right) \leq -\alpha. \quad (45)$$

To see this, by Lemma 9, there exists a $c \in \mathbb{R}^+$ such that

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{|V_n|} \log \mathbb{P} \left(\hat{\mu}^n(\mathcal{Y}^n) \notin \mathcal{A}_c \right) \leq -\alpha. \quad (46)$$

But by Lemma 10, there exists an $m \geq c$ such that if $\hat{\mu}^n(\mathcal{Y}^n) \in \mathcal{A}_c$, then $d^{\mathcal{P}}(\dot{\Psi}^m(\hat{\mu}^n(\mathcal{Y}^n)), \tilde{\Psi}^n(\hat{\mu}^n(\mathcal{Y}^n))) \leq \delta$. This establishes (45).

To apply the result in [22, Theorem 4.2.23], we also require that for each $\alpha \in \mathbb{R}^+$, writing $\mathcal{A}_\alpha^* = \{\mu \in \mathcal{P}_{\bar{S}}(\bar{\mathcal{T}}^{\mathbb{Z}^d}) | I_{\mathcal{Y}}(\mu) \leq \alpha\}$,

$$\lim_{m \rightarrow \infty} \sup_{\mu \in \mathcal{A}_\alpha^*} d^{\mathcal{P}}(\dot{\Psi}(\mu), \dot{\Psi}^m(\mu)) = 0. \quad (47)$$

Thanks to Lemma 11, it suffices to prove that for any $c \in \mathbb{R}^+$,

$$\lim_{m \rightarrow \infty} \sup_{\mu \in \mathcal{A}_c} d^{\mathcal{P}}(\dot{\Psi}(\mu), \dot{\Psi}^m(\mu)) = 0. \quad (48)$$

Now once $m \geq c$, $\dot{\Psi}^m(\mu) = \mu \circ (\Psi^m)^{-1}$, and since $\dot{\Psi}(\mu) = \lim_{m \rightarrow \infty} \dot{\Psi}^m(\mu)$ (thanks to Lemma 10),

$$\lim_{m \rightarrow \infty} \sup_{\mu \in \mathcal{A}_c} d^{\mathcal{P}}(\dot{\Psi}(\mu), \dot{\Psi}^m(\mu)) \leq 2 \lim_{m \rightarrow \infty} \sup_{n \geq m} \sup_{\mu \in \mathcal{A}_c} d^{\mathcal{P}}(\dot{\Psi}^m(\mu), \tilde{\Psi}^n(\mu)).$$

(48) follows as a consequence of the above and Lemma 10.

The theorem now follows from the fact that the laws $(\Pi_{\mathcal{Y}}^n)_{n \in \mathbb{Z}^+}$ satisfy an LDP with good rate function, $\dot{\Psi}^m$ is continuous, (45), (47) and [22, Theorem 4.2.23].

The following lemma notes that Ψ^n preserves \bar{V}_n -periodicity.

Lemma 1 For each $W^n \in \mathcal{T}^{V_n}$, there exists a unique solution U to (3). This solution satisfies, for all $p \in \mathbb{Z}^d$,

$$\Psi^n(\tilde{\mathcal{Y}}^n)^p = U^p \bmod V_n.$$

Furthermore

$$\hat{\mu}^n(U) = \hat{\mu}^n(\mathcal{Y}^n) \circ (\Psi^n)^{-1}. \quad (49)$$

Proof The existence and uniqueness of the solution (3) follows from Lemma 2. It follows from the definition that for all $j \in \mathbb{Z}^d$, $\Psi^n(S^j \tilde{\mathcal{Y}}^n) = S^j \Psi(\tilde{\mathcal{Y}}^n)$. (49) follows directly from this and the definition of the empirical measure.

Proof (Proof of Corollary 1) The assumptions mean that $\mu_{\mathcal{Y}}$ is the unique probability measure such that $I_{\mathcal{Y}}(\mu_{\mathcal{Y}}) = 0$. This means, in turn, that $\tilde{\Psi}(\mu_{\mathcal{Y}})$ is the unique zero of I . It can then be shown using the Borel-Cantelli Lemma that $\hat{\mu}^n(U) \rightarrow \tilde{\Psi}(\mu_{\mathcal{Y}})$ almost surely (see for instance [28, Theorem II.6.4]).

5 Existence and continuity of $\tilde{\Psi}^m$

The map Ψ^m (as defined in (41)) is not well-defined on all of $\bar{\mathcal{T}}^{\mathbb{Z}^d}$, and it is not necessarily continuous on closed subsets either. Our first task therefore is to define Ψ^m on a weighted subspace $\bar{\mathcal{T}}_{\lambda_m}^{\mathbb{Z}^d}$ of $\bar{\mathcal{T}}^{\mathbb{Z}^d}$ (which we outline in further detail below). We then show that the induced map $\mu \rightarrow \mu \circ (\Psi^m)^{-1}$ is continuous on the closed sets $\mathcal{A}_c \subset \mathcal{P}_{\bar{\mathcal{S}}}(\bar{\mathcal{T}}^{\mathbb{Z}^d})$. The methods of this section are very similar to [32]. The integer m is fixed throughout this section.

We start by defining the weights $\{\lambda_m^j\}_{j \in \mathbb{Z}^d}$ which we use to modulate the convergence (similar methods have been used previously in, for example, [61] and [32]). For $\theta \in [-\pi, \pi]^d$, let $\kappa_m^k = \mathbf{1}_{k \in V_m}$, $\tilde{\kappa}_m(\theta) = \sum_{k \in \mathbb{Z}^d} \kappa_m^k \exp(-i\langle \theta, k \rangle)$ and

$$\tilde{\lambda}_m(\theta) = h \left(\rho |V_m| - \tilde{\kappa}_m(\theta) \right)^{-1},$$

where we recall that $\rho > 1$ is used in the definition of \mathcal{A}_c in (39) and h is chosen to ensure that

$$\frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \tilde{\lambda}_m(\theta) d\theta = 1. \quad (50)$$

Let $\lambda_m^j = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \exp(i\langle \theta, j \rangle) \tilde{\lambda}_m(\theta) d\theta$. In the case that $\rho = 2$, it was proved in [32, Lemma 5] that

$$\sum_{j \in \mathbb{Z}^d} \lambda_m^j = 1, \quad (51)$$

$$\lambda_m^j > 0, \quad (52)$$

$$\sum_{k \in V_m} \lambda_m^{j-k} \kappa_m^k \leq \rho |V_m| \lambda_m^j. \quad (53)$$

The above identities easily generalise to any $\rho > 1$. An immediate consequence of (53) is, for any non-negative $(v^j)_{j \in \mathbb{Z}^d} \subset \mathbb{R}$,

$$\sum_{j \in \mathbb{Z}^d, k \in V_m} \lambda_m^j v^{j+k} \leq \rho |V_m| \sum_{j \in \mathbb{Z}^d} \lambda_m^j v^j, \quad (54)$$

as long as the above sums are finite. Let $\bar{\mathcal{T}}_{\lambda_m}^{\mathbb{Z}^d}$ be the space of all $\mathcal{Z} := (R^j, (\omega^{j,k})_{k \in \mathbb{Z}^d})_{j \in \mathbb{Z}^d} \in \bar{\mathcal{T}}^{\mathbb{Z}^d}$ such that

$$\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|R^j\|_T^2 < \infty \quad (55)$$

$$\sum_{j \in \mathbb{Z}^d} \lambda_m^j \sum_{k \in \mathbb{Z}^d} |\omega^{j,k}|_{\mathfrak{C}}^2 < \infty. \quad (56)$$

Define $\Psi^m : \bar{\mathcal{T}}_{\lambda_m}^{\mathbb{Z}^d} \rightarrow \mathcal{T}^{\mathbb{Z}^d}$ as follows. Using the above definitions, for $\mathcal{Z} \in \bar{\mathcal{T}}^{\mathbb{Z}^d}$, with $\mathcal{Z}^j := (R^j, (\omega^{j,l})_{l \in \mathbb{Z}^d})$, we define $\Psi^m(\mathcal{Z}) := X = (X_t^j)_{j \in \mathbb{Z}^d, t \in [0, T]}$ to be such that for any $t \in [0, T]$ and $j \in \mathbb{Z}^d$,

$$X_t^j := \int_0^t \left(\mathfrak{b}_s(X^j) + \sum_{k \in V_m} \Lambda_s^k(\omega^{j,k}, X^j, X^{(j+k) \bmod V_m}) \right) ds + R_t^j. \quad (57)$$

Lemma 2 *For each $\mathcal{Z} \in \bar{\mathcal{T}}_{\lambda_m}^{\mathbb{Z}^d}$, $\Psi^m(\mathcal{Z})$ exists and is unique.*

Proof Fix $\mathcal{Z} = (R^j, \omega^{j,k})_{j,k \in \mathbb{Z}^d} \in \bar{\mathcal{T}}_{\lambda_m}^{\mathbb{Z}^d}$. We assume that $\omega^{j,k} = \mathfrak{N}$ if $j - k \notin V_m$, because if $\Psi^m(\mathcal{Z})$ were to exist then it would be independent of these values. We prove the existence in two steps. We start by finding a series of periodic approximations $\mathcal{Z}(p)$ of \mathcal{Z} . $\Psi^m(\mathcal{Z}(p))$ can be shown to exist through the Cauchy-Peano existence theorem, and then it will be shown that $\{\Psi^m(\mathcal{Z}(p))\}_{p \in \mathbb{Z}^+}$ is Cauchy, yielding the lemma. The proof is very similar to Lemmas 6-8 of [32] and we therefore do not provide every detail.

For $p \in \mathbb{Z}^+$, let $\mathcal{Z}(p) \in \bar{\mathcal{T}}^{\mathbb{Z}^d}$ have the property that, writing $\mathcal{Z}(p)^j = (R(p)^j, (\beta(p)^{j,k})_{k \in \mathbb{Z}^d})$ $R(p)^j = R(p)^a$, $\beta(p)^{j,k} = \beta(p)^{a,k}$, where $a = j \bmod V_p$. Very similarly to [32, Lemma 6], we can choose $\{\mathcal{Z}(p)\}_{p \in \mathbb{Z}^+}$ such that

$$\lim_{p \rightarrow \infty} \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|R(p)^j - R^j\|_T = 0 \quad (58)$$

$$\lim_{p \rightarrow \infty} \sum_{j \in \mathbb{Z}^d, k \in V_m} \lambda_m^j d_{\mathfrak{C}}(\beta(p)^{j,k}, \beta^{j,k})^2 = 0. \quad (59)$$

Now $\Psi^m(\mathcal{Z}(p))$ must exist for any $p \in \mathbb{Z}^+$. In brief, this is because for any $r \in \mathbb{Z}^d$, if the solution were to exist then $S^r \Psi^m(\mathcal{Z}(p)) = \Psi^m(S^r \mathcal{Z}(p))$. We can therefore reduce the existence of $\Psi^m(\mathcal{Z}(p))$ to the existence of a solution to a finite-dimensional ordinary differential equation indexed over V_p , as in [32, Lemma 7]. The solution $\Psi^m(\mathcal{Z}(p))$ exists for any $p \in \mathbb{Z}^+$ thanks to the

generalised form of the Cauchy-Peano existence theorem in [32, Lemma 8].
Now, by Lemma 6,

$$\begin{aligned} \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|\Psi^m(\mathcal{Z}(q))^j - \Psi^m(\mathcal{Z}(p))^j\|_T &\leq \exp(TC + (1 + \rho)TC_J|V_m|) \times \\ &\left[2 \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|R(p)^j - R(q)^j\|_T + T(1 + \sqrt{\rho})|V_m|^{\frac{1}{2}} \left(\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|\Psi^m(\mathcal{Z}(p))^j\|_T^2 \right)^{\frac{1}{2}} \times \right. \\ &\quad \left. \left(\sum_{j \in \mathbb{Z}^d, k \in V_m} \lambda_m^j d_{\mathfrak{C}}(\beta(p)^{j,k}, \beta(q)^{j,k})^2 \right)^{\frac{1}{2}} \right]. \end{aligned}$$

By Lemma 7, since $(A + B)^2 \leq 2A^2 + 2B^2$ and $|\cdot|_{\mathfrak{C}} \leq C_J$,

$$\|\Psi^m(\mathcal{Z}(p))^j\|_T^2 \leq 2 \left[\exp\left(2T(C + 2C_J|V_m|)\right) + 4 \exp\left(2T(C + C_J|V_m|)\right) \|R(p)^j\|_T^2 \right].$$

This means that

$$\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|\Psi^m(\mathcal{Z}(p))^j\|_T^2 \leq 2 \exp\left(2T(C + 2C_J|V_m|)\right) \left(1 + 4 \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|R(p)^j\|_T^2\right).$$

Since

$$\lim_{p \rightarrow \infty} \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|R(p)^j\|_T^2 = \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|R^j\|_T^2,$$

there must exist a uniform upper bound L such that

$\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|\Psi^m(\mathcal{Z}(p))^j\|_T^2 \leq L$ for all $p \in \mathbb{Z}^+$. It is therefore a consequence of (58)-(59) that

$$\lim_{q \rightarrow \infty} \sup_{p \geq q} \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|\Psi^m(\mathcal{Z}(q))^j - \Psi^m(\mathcal{Z}(p))^j\|_T = 0.$$

Let

$$\Psi^m(\mathcal{Z})^j = \lim_{p \rightarrow \infty} \Psi^m(\mathcal{Z}(p))^j.$$

To show that $\Psi^m(\mathcal{Z})$ satisfies (57), we must verify that

$$\begin{aligned} &\int_0^t \sum_{k \in V_m} \Lambda_s^k(\beta(p)^{j,k}, \Psi^m(\mathcal{Z}(p))^j, \Psi^m(\mathcal{Z}(p))^{j+k}) ds \\ &\rightarrow \int_0^t \sum_{k \in V_m} \Lambda_s^k(\beta^{j,k}, \Psi^m(\mathcal{Z})^j, \Psi^m(\mathcal{Z})^{j+k}) ds \\ &\int_0^t \mathfrak{b}_s(\Psi^m(\mathcal{Z}(p))^j) ds \rightarrow \int_0^t \mathfrak{b}_s(\Psi^m(\mathcal{Z})^j) ds. \end{aligned}$$

In fact the above identities follow from Assumptions 1 and 2 and the dominated convergence theorem. The uniqueness of $\Psi^m(\mathcal{Z})$ follows from Lemma 6.

Recall that $\tilde{\Psi}^m(\mu) := \mu \circ (\Psi^m)^{-1}$. The following lemma notes that this is well-defined for $\mu \in \mathcal{A}_c$ (\mathcal{A}_c is defined in (39)).

Lemma 3 $\tilde{\Psi}^m : \mathcal{A}_c \rightarrow \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}^d})$ is well-defined and uniformly continuous. That is, for all ϵ there exists a δ such that for all $\mu, \nu \in \mathcal{A}_c$ satisfying $\bar{d}^{\mathcal{P}}(\mu, \nu) \leq \delta$, $d^{\mathcal{P}}(\tilde{\Psi}^m(\mu), \tilde{\Psi}^m(\nu)) \leq \epsilon$. Furthermore,

$$\mathbb{E}^\mu \left[\|\Psi^m(\mathcal{Z})^j\|_T^2 \right] < \infty, \quad (60)$$

the above expectation being the same for all $j \in \mathbb{Z}^d$ since $\tilde{\Psi}^m(\mu)$ is stationary.

Proof We first observe that if $\mu \in \mathcal{A}_c$, and μ is the law of $\{\mathcal{Z}^j\}_{j \in \mathbb{Z}^d}$, where $\mathcal{Z}^j = (R^j, (\omega^{j,k})_{k \in \mathbb{Z}^d})$, then since

$$\mathbb{E}^\mu \left[\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|R^j\|_T^2 \right] \leq c \sum_{j \in \mathbb{Z}^d} \lambda_m^j < \infty,$$

$\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|R^j\|_T^2 < \infty$, μ -almost surely. Similarly, since $\mathbb{E}^\mu \left[\sum_{k \in \mathbb{Z}^d} |\omega^{j,k}|_{\mathcal{C}}^2 \right] < \infty$ (by the stationarity of μ , this integral is independent of j), it must be that $\sum_{j \in \mathbb{Z}^d, k \in \mathbb{Z}^d} \lambda_m^j |\omega^{j,k}|_{\mathcal{C}}^2 < \infty$, μ -almost surely. Thanks to Lemma 2, this means that $\Psi^m(\mathcal{Z})$ exists for μ almost-every \mathcal{Z} , and therefore that $\tilde{\Psi}^m(\mu)$ is well-defined.

The bound in Lemma 7 implies (60), since $\mathbb{E} \left[\|R^j\|_T^2 \right] \leq c$.

Now it is an immediate consequence of the definition of the metric in (2) that for any $s \in \mathbb{Z}^+$,

$$\lim_{\gamma \rightarrow 0} \sup_{\mu, \nu \in \mathcal{A}_c, \bar{d}^{\mathcal{P}}(\mu, \nu) \leq \gamma} d^{\mathcal{P}} \left(\nu \circ (\Psi^m)^{-1}, \mu \circ (\Psi^m)^{-1} \right) \leq 2^{-s-1} + \sum_{r=1}^s \lim_{\gamma \rightarrow 0} \sup_{\mu, \nu \in \mathcal{A}_c, \bar{d}^{\mathcal{P}}(\mu, \nu) \leq \gamma} d_r^{\mathcal{P}} \left(\pi_{V_r}^{\mathcal{P}}(\nu \circ (\Psi^m)^{-1}), \pi_{V_r}^{\mathcal{P}}(\mu \circ (\Psi^m)^{-1}) \right). \quad (61)$$

It thus suffices for us to prove that for an arbitrary $r \in \mathbb{Z}^+$,

$$\lim_{\gamma \rightarrow 0} \sup_{\mu, \nu \in \mathcal{A}_c, \bar{d}^{\mathcal{P}}(\mu, \nu) \leq \gamma} d_r^{\mathcal{P}} \left(\pi_{V_r}^{\mathcal{P}}(\nu \circ (\Psi^m)^{-1}), \pi_{V_r}^{\mathcal{P}}(\mu \circ (\Psi^m)^{-1}) \right) = 0. \quad (62)$$

Let \mathcal{B}^{V_r} be the set of all V_r cylinder sets, that is every $B \in \mathcal{B}^{V_r} \subset \mathcal{B}(\mathcal{T}^{\mathbb{Z}^d})$ is such that $B = (\pi^{V_r})^{-1} B_2$ for some $B_2 \in \mathcal{B}(\mathcal{T}^{V_r})$. For $\delta > 0$, let $B^\delta \subset B$ be the δ -interior relative to the norm on \mathcal{T}^{V_r} , i.e. such that

$$B^\delta = \left\{ X \in \mathcal{T}^{\mathbb{Z}^d} \mid \left\{ Y \in \mathcal{T}^{\mathbb{Z}^d} \mid \sum_{j \in V_r} \|X^j - Y^j\|_T \leq \delta \right\} \subseteq B \right\}. \quad (63)$$

Let $\bar{B} = (\Psi^m)^{-1}(B)$ and $\bar{B}^\delta = (\Psi^m)^{-1}(B^\delta)$. We use Lemma 4 to establish (62). This lemma implies that for any δ there exists a γ such that

$$\sup_{\mu, \nu \in \mathcal{A}_c, \bar{d}^{\mathcal{P}}(\mu, \nu) \leq \gamma} \sup_{B \in \mathcal{B}^{V_r}} (\nu(\bar{B}) - \mu(\bar{B}^\delta)) \leq \delta. \quad (64)$$

It follows from the above equation and the definition of the Prokhorov metric that

$$\sup_{\mu, \nu \in \mathcal{A}_c, \bar{d}^{\mathcal{P}}(\mu, \nu) \leq \gamma} d_r^{\mathcal{P}} \left(\pi_{V_r}^{\mathcal{P}}(\nu \circ (\Psi^m)^{-1}), \pi_{V_r}^{\mathcal{P}}(\mu \circ (\Psi^m)^{-1}) \right) \leq \delta. \quad (65)$$

Since δ is arbitrary, (62) is an immediate consequence of this, which completes the proof.

The variables in the following lemma are defined in the proof of Lemma 3.

Lemma 4 For any $r \in \mathbb{Z}^+$ and $\delta > 0$,

$$\lim_{\gamma \rightarrow 0} \sup_{\mu, \nu \in \mathcal{A}_c, \bar{d}^{\mathcal{P}}(\mu, \nu) \leq \gamma} \sup_{B \in \mathcal{B}^{V_r}} (\nu(\bar{B}) - \mu(\bar{B}^\delta)) \leq \delta. \quad (66)$$

Proof For $\epsilon > 0$ and $q > m$, let $\mathcal{H}_q^\epsilon \subset \bar{\mathcal{T}}_{\lambda_m}^{\mathbb{Z}^d}$ be the subset of all $\mathcal{Z} = (R^j, \beta^{j,k})_{j,k \in \mathbb{Z}^d}$ such that, writing $Z = \Psi^m(\mathcal{Z})$,

$$\sum_{j \notin V_q} \lambda_m^j \|R^j\|_T \leq \epsilon, \quad (67)$$

$$\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|Z^j\|_T^2 \leq \epsilon^{-1}, \quad (68)$$

$$\sum_{j \notin V_q, k \in V_m} \lambda_m^j |\beta^{j,k}|_{\mathfrak{C}}^2 \leq \epsilon^2. \quad (69)$$

Writing $\bar{B}_q^\epsilon = \bar{B} \cap \mathcal{H}_q^\epsilon$ and $\bar{B}_q^{\delta, \epsilon} = \bar{B}^\delta \cap \mathcal{H}_q^\epsilon$, it may be observed that

$$\nu(\bar{B}) - \mu(\bar{B}^\delta) \leq \nu(\bar{B}_q^\epsilon) - \mu(\bar{B}_q^{\delta, \epsilon}) + \nu((\mathcal{H}_q^\epsilon)^c). \quad (70)$$

Suppose that $\mathcal{X}, \mathcal{Z} \in \mathcal{H}_q^\epsilon$, with $\mathcal{X} := (Q^j, \omega^{j,k})_{j,k \in \mathbb{Z}^d}$, $\mathcal{Z} := (R^j, \beta^{j,k})_{j,k \in \mathbb{Z}^d} \in \bar{\mathcal{T}}_{\lambda}^{\mathbb{Z}^d}$. Let $X = \Psi^m(\mathcal{X})$ and $Z = \Psi^m(\mathcal{Z})$. By Lemma 6, for any $r \in \mathbb{Z}^+$,

$$\sum_{j \in V_r} \lambda_m^j \|X^j - Z^j\|_T \leq A_m \left(H_1^q + H_2^q \right),$$

where $A_m = T \exp(TC + T(1 + \rho)C_J|V_m|)$,

$$\begin{aligned} H_1^q &= 2 \sum_{j \in V_q} \lambda_m^j \|Q^j - R^j\|_T \\ &+ (1 + \sqrt{\rho})|V_m|^{\frac{1}{2}} \left(\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|Z^j\|_s^2 \right)^{\frac{1}{2}} \left(\sum_{j \in V_q, k \in V_m} \lambda_m^j d_{\mathfrak{C}}(\omega^{j,k}, \beta^{j,k})^2 \right)^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned}
H_2^q &= 2 \sum_{j \notin V_q} \lambda_m^j \|Q^j - R^j\|_T \\
&\quad + (1 + \sqrt{\rho}) |V_m|^{\frac{1}{2}} \left(\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|Z^j\|_s^2 \right)^{\frac{1}{2}} \left(\sum_{j \notin V_q, k \in V_m} \lambda_m^j d_{\mathfrak{C}}(\omega^{j,k}, \beta^{j,k})^2 \right)^{\frac{1}{2}} \\
&\leq 2 \sum_{j \notin V_q} \lambda_m^j [\|Q^j\|_T + \|R^j\|_T] \\
&\quad + (1 + \sqrt{\rho}) \sqrt{2} |V_m|^{\frac{1}{2}} \left(\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|Z^j\|_s^2 \right)^{\frac{1}{2}} \left(\sum_{j \notin V_q, k \in V_m} \lambda_m^j (|\omega^{j,k}|_{\mathfrak{C}}^2 + |\beta^{j,k}|_{\mathfrak{C}}^2) \right)^{\frac{1}{2}},
\end{aligned}$$

recalling that $|\cdot|_{\mathfrak{C}} = d_{\mathfrak{C}}(\cdot, \mathfrak{N})$. We have used the inequalities $\sqrt{F+G} \leq \sqrt{F} + \sqrt{G}$, and $(F+G)^2 \leq 2F^2 + 2G^2$. Using the definition of \mathcal{H}_q^ϵ , and also the Cauchy-Schwarz Inequality, we find that

$$\begin{aligned}
H_2^q &\leq 4\epsilon + \sqrt{2}(1 + \sqrt{\rho}) |V_m|^{\frac{1}{2}} \epsilon^{-\frac{1}{2}} \epsilon \\
&= 4\epsilon + \sqrt{2}(1 + \sqrt{\rho}) |V_m|^{\frac{1}{2}} \epsilon^{\frac{1}{2}}.
\end{aligned}$$

Take ϵ to be sufficiently small that $H_2^q \leq \frac{\delta}{2A_m}$ and

$$4c \left(\frac{1}{2} \epsilon^{-1} \exp(-2T(C + 2C_J |V_m|)) - 1 \right)^{-1} \leq \frac{\delta}{4}. \quad (71)$$

We find that

$$\sum_{j \in V_r} \lambda_m^j \|X^j - Z^j\|_T \leq A_m \bar{d}_q(\mathcal{X}, \mathcal{Z}) \left(2 + (1 + \sqrt{\rho}) |V_m|^{\frac{1}{2}} \epsilon^{-\frac{1}{2}} \right) + \frac{\delta}{2}. \quad (72)$$

By Lemma 5, and noting (71), we may take q to be sufficiently large that for all $\nu \in \mathcal{A}_c$,

$$\nu \left((\mathcal{H}_q^\epsilon)^c \right) \leq \frac{\delta}{2}. \quad (73)$$

The lemma will follow once we have established the following claim.

Claim:

We claim that for γ sufficiently small,

$$\sup_{\mu, \nu \in \mathcal{A}_c, \bar{d}^{\mathcal{P}}(\mu, \nu) \leq \gamma} \sup_{B \in \mathcal{B}_m} \nu(\bar{B}) - \mu(\bar{B}^\delta) \leq 2^q \gamma + \frac{\delta}{2}. \quad (74)$$

Thanks to (70) and (73), it suffices to prove that, for γ sufficiently small,

$$\sup_{\mu, \nu \in \mathcal{A}_c, \bar{d}^{\mathcal{P}}(\mu, \nu) \leq \gamma} \sup_{B \in \mathcal{B}_m} \nu(\bar{B}_q^\epsilon) - \mu(\bar{B}_q^{\delta, \epsilon}) \leq \gamma 2^q.$$

For $\eta > 0$, let $\tilde{B}_q^{\eta, \epsilon} \subset \bar{B}_q^\epsilon$ be the η relative interior, that is

$$\tilde{B}_q^{\eta, \epsilon} = \left\{ \mathcal{X} \in \bar{B}_q^\epsilon \mid \{ \mathcal{W} \in \bar{\mathcal{T}}^{\mathbb{Z}^d} \mid \bar{d}_q(\mathcal{X}, \mathcal{W}) \leq \eta \} \subseteq \bar{B}_q^\epsilon \right\}. \quad (75)$$

It follows from (72) that if

$$\eta \leq \frac{\delta}{2} \left(A_m (2 + (1 + \sqrt{\rho}) |V_m|^{\frac{1}{2}} \epsilon^{-\frac{1}{2}}) \right)^{-1}, \quad (76)$$

then

$$\tilde{B}_q^{\eta, \epsilon} \subseteq \bar{B}_q^{\delta, \epsilon}.$$

This means that if (76) is satisfied, then

$$\begin{aligned} \nu(\bar{B}_q^\epsilon) - \mu(\bar{B}_q^{\delta, \epsilon}) &\leq \nu(\bar{B}_q^\epsilon) - \mu(\tilde{B}_q^{\eta, \epsilon}) \\ &\leq \bar{d}_q^P(\nu, \mu) \text{ if } \bar{d}_q^P(\nu, \mu) \leq \eta, \end{aligned}$$

thanks to the definition of the Prokhorov Metric. Now it follows from the definition that if $\bar{d}^P(\mu, \nu) \leq \gamma$, then $\bar{d}_q^P(\nu, \mu) \leq 2^q \gamma$. We have thus established our claim in (74).

Lemma 5 For all $\mu \in \mathcal{A}_c$, and q sufficiently large,

$$\mu\left((\mathcal{H}_q^\epsilon)^c\right) \leq \sqrt{c}\epsilon^{-1} \sum_{j \notin V_q} \lambda_m^j + cL(m, \epsilon), \quad (77)$$

where $L(m, \epsilon)$ is defined in (78) and \mathcal{H}_q^ϵ is defined in (67) -(69).

Proof We suppose that μ is the law of $\mathcal{Z} = (R^j, \beta^{j, k})_{j, k \in \mathbb{Z}^d}$ and write $Z = \Psi^m(\mathcal{Z})$. We obtain bounds on each of (67) -(69). By Chebyshev's Inequality,

$$\begin{aligned} \mu\left(\sum_{j \notin V_q} \lambda_m^j \|R^j\|_T \geq \epsilon\right) &\leq \epsilon^{-1} \mathbb{E}^\mu \left[\sum_{j \notin V_q} \lambda_m^j \|R^j\|_T \right] \\ &\leq \epsilon^{-1} \sum_{j \notin V_q} \lambda_m^j \mathbb{E}^\mu \left[\|R^j\|_T^2 \right]^{\frac{1}{2}} \\ &\leq \sqrt{c}\epsilon^{-1} \sum_{j \notin V_q} \lambda_m^j, \end{aligned}$$

using the definition of \mathcal{A}_c in (39). By Lemma 7, the bound $|\cdot|_\epsilon \leq C_J$ of Assumption 2, and the fact that $(A + B)^2 \leq 2(A^2 + B^2)$, for each $j \in \mathbb{Z}^d$,

$$\|Z^j\|_T^2 \leq 2(1 + 4 \|R^j\|_T^2) \exp(2T(C + 2C_J |V_m|)),$$

and therefore

$$\begin{aligned} \|R^j\|_T^2 &\geq \frac{1}{4} \left(\frac{1}{2} \|Z^j\|_T^2 \exp(-2T(C + 2C_J |V_m|)) - 1 \right) \\ \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|R^j\|_T^2 &\geq \frac{1}{4} \left(\frac{1}{2} \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|Z^j\|_T^2 \exp(-2T(C + 2C_J |V_m|)) - 1 \right), \end{aligned}$$

after summing over j . Hence, writing

$$L(m, \epsilon) = 4 \left(\frac{1}{2} \epsilon^{-1} \exp(-2T(C + 2C_J|V_m|)) - 1 \right)^{-1}, \quad (78)$$

and noting that $\sum_{j \in \mathbb{Z}^d} \lambda_m^j = 1$,

$$\begin{aligned} & \mu \left[\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|Z^j\|_T^2 \geq \epsilon^{-1} \right] \\ & \leq \mu \left[\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|R^j\|_T^2 \geq \frac{1}{4} \left(\frac{1}{2} \epsilon^{-1} \exp(-2T(C + 2C_J|V_m|)) - 1 \right) \right] \\ & \leq L(m, \epsilon) \mathbb{E}^\mu \left[\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|R^j\|_T^2 \right] \\ & \leq cL(m, \epsilon), \end{aligned}$$

where we have made use of Chebyshev's Inequality. Finally, observe that thanks to Assumption 2,

$$\sum_{j \notin V_q} \lambda_m^j \sum_{k \in V_m} |\beta^{j,k}|_{\mathfrak{C}}^2 \leq \sum_{j \notin V_q} \lambda_m^j |V_m| C_J^2.$$

Since $\sum_{j \in \mathbb{Z}^d} \lambda_m^j = 1$, for q sufficiently large the above must be less than or equal to ϵ^2 , so that (69) is satisfied.

Lemma 6 Suppose that $\mathcal{X} = (Q^j, \omega^{j,k})_{j,k \in \mathbb{Z}^d} \in \bar{\mathcal{T}}_{\lambda_m}^{\mathbb{Z}^d}$ and $\mathcal{Z} = (R^j, \beta^{j,k})_{j,k \in \mathbb{Z}^d} \in \bar{\mathcal{T}}_{\lambda_m}^{\mathbb{Z}^d} \cap \bar{\mathcal{T}}_{\lambda_n}^{\mathbb{Z}^d}$. Suppose that for $n \geq m$ there exist solutions $X = \Psi^m(\mathcal{X})$ and $Z = \Psi^n(\mathcal{Z})$ to (57). Then

$$\begin{aligned} & \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|X^j - Z^j\|_T \leq \exp(TC + T(1 + \rho)C_J|V_m|) \\ & \times \left[(1 + \sqrt{\rho})T|V_m|^{\frac{1}{2}} \left(\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|Z^j\|_T^2 \right)^{\frac{1}{2}} \left(\sum_{j \in \mathbb{Z}^d, k \in V_m} \lambda_m^j d_{\mathfrak{C}}(\omega^{j,k}, \beta^{j,k})^2 \right)^{\frac{1}{2}} \right. \\ & \left. + 2 \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|Q^j - R^j\|_T + T \sum_{j \in \mathbb{Z}^d, k \in V_n - V_m} \lambda_m^j |\beta^{j,k}|_{\mathfrak{C}} (\|Z^j\|_T + 1) \right]. \end{aligned}$$

Proof Observe that

$$\begin{aligned}
X_t^j - Z_t^j &= \int_0^t \left(\mathfrak{b}_s(X^j) - \mathfrak{b}_s(Z^j) - \sum_{k \in V_n - V_m} \Lambda_s^k(\beta^{j,k}, Z^j, Z^{j+k}) \right. \\
&\quad \left. + \sum_{k \in V_m} \Lambda_s^k(\omega^{j,k}, X^j, X^{j+k}) - \Lambda_s^k(\beta^{j,k}, Z^j, Z^{j+k}) \right) ds + Q_t^j - R_t^j \\
&= \int_0^t \left(\mathfrak{b}_s(X^j) - \mathfrak{b}_s(Z^j) - \sum_{k \in V_n - V_m} \Lambda_s^k(\beta^{j,k}, Z^j, Z^{j+k}) \right. \\
&\quad \left. + \sum_{k \in V_m} \left[\Lambda_s^k(\omega^{j,k}, X^j, X^{j+k}) - \Lambda_s^k(\omega^{j,k}, Z^j, X^{j+k}) \right. \right. \\
&\quad \left. \left. + \Lambda_s^k(\omega^{j,k}, Z^j, X^{j+k}) - \Lambda_s^k(\omega^{j,k}, Z^j, Z^{j+k}) \right. \right. \\
&\quad \left. \left. + \Lambda_s^k(\omega^{j,k}, Z^j, Z^{j+k}) - \Lambda_s^k(\beta^{j,k}, Z^j, Z^{j+k}) \right] \right) ds + Q_t^j - R_t^j.
\end{aligned}$$

Now suppose that $[\tau, \gamma]$ are such that $X_t^j \geq Z_t^j$ for all $t \in [\tau, \gamma]$. Applying the inequalities of Assumptions 1 and 2, we find that

$$\begin{aligned}
X_t^j - Z_t^j &\leq X_\tau^j - Z_\tau^j + \int_\tau^t \left(C \|X^j - Z^j\|_s + \sum_{k \in V_n - V_m} |\beta^{j,k}|_{\mathfrak{C}} (\|Z^j\|_s + 1) \right. \\
&\quad \left. C_J |V_m| \|X^j - Z^j\|_s + C_J \sum_{k \in V_m} \|X^{j+k} - Z^{j+k}\|_s \right. \\
&\quad \left. + \sum_{k \in V_m} d_{\mathfrak{C}}(\omega^{j,k}, \beta^{j,k}) (\|Z^j\|_s + \|Z^{j+k}\|_s) \right) ds + Q_t^j - R_t^j - Q_\tau^j + R_\tau^j.
\end{aligned}$$

We take τ to be such that, either $\tau = 0$, or $X_t^j < Z_t^j$ as $t \rightarrow \tau^-$. Now since if $\tau = 0$, then $X_\tau^j - Z_\tau^j = Q_0^j - R_0^j$, whichever of these cases holds we find that for all $t \in [\tau, \gamma]$,

$$\begin{aligned}
|X^j - Z^j|_t &\leq 2 \|Q^j - R^j\|_T + \int_\tau^t \left(C \|X^j - Z^j\|_s + C_J |V_m| \|X^j - Z^j\|_s \right. \\
&\quad \left. + C_J \sum_{k \in V_m} \|X^{j+k} - Z^{j+k}\|_s + \sum_{k \in V_m} d_{\mathfrak{C}}(\omega^{j,k}, \beta^{j,k}) (\|Z^j\|_s + \|Z^{j+k}\|_s) \right. \\
&\quad \left. + \sum_{k \in V_n - V_m} |\beta^{j,k}|_{\mathfrak{C}} (\|Z^j\|_s + 1) \right) ds \\
&\leq 2 \|Q^j - R^j\|_T + \int_0^t \left((C + C_J |V_m|) \|X^j - Z^j\|_s + C_J \sum_{k \in V_m} \|X^{j+k} - Z^{j+k}\|_s \right. \\
&\quad \left. + \|Z^j\|_s \sum_{k \in V_m} d_{\mathfrak{C}}(\omega^{j,k}, \beta^{j,k}) + \left(\sum_{k \in V_m} \|Z^{j+k}\|_s^2 \right)^{\frac{1}{2}} \left(\sum_{k \in V_m} d_{\mathfrak{C}}(\omega^{j,k}, \beta^{j,k})^2 \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \sum_{k \in V_n - V_m} |\beta^{j,k}|_{\mathfrak{C}} (\|Z^j\|_s + 1) \right) ds.
\end{aligned}$$

We could demonstrate the same inequality if $[\tau, \gamma]$ were such that $X_t^j \leq Z_t^j$ for all $t \in [\tau, \gamma]$.

We thus find that, making use of the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|X^j - Z^j\|_t \leq 2 \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|Q^j - R^j\|_T \\
& + \int_0^t \left[(C + C_J |V_m|) \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|X^j - Z^j\|_s + C_J \sum_{j \in \mathbb{Z}^d, k \in V_m} \lambda_m^j \|X^{j+k} - Z^{j+k}\|_s \right. \\
& + \left(\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|Z^j\|_s^2 \right)^{\frac{1}{2}} \left(\sum_{j \in \mathbb{Z}^d} \lambda_m^j \left(\sum_{k \in V_m} d_{\mathfrak{C}}(\omega^{j,k}, \beta^{j,k}) \right)^2 \right)^{\frac{1}{2}} \\
& + \left(\sum_{j \in \mathbb{Z}^d, k \in V_m} \lambda_m^j \|Z^{j+k}\|_s^2 \right)^{\frac{1}{2}} \left(\sum_{j \in \mathbb{Z}^d, k \in V_m} \lambda_m^j d_{\mathfrak{C}}(\omega^{j,k}, \beta^{j,k})^2 \right)^{\frac{1}{2}} \\
& + \sum_{j \in \mathbb{Z}^d, k \in V_n - V_m} \lambda_m^j |\beta^{j,k}|_{\mathfrak{C}} (\|Z^j\|_s + 1) \Big] ds \\
& \leq 2 \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|Q^j - R^j\|_T \\
& + \int_0^t \left[(C + C_J |V_m|) \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|X^j - Z^j\|_s + \rho C_J |V_m| \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|X^j - Z^j\|_s \right. \\
& + |V_m|^{\frac{1}{2}} \left(\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|Z^j\|_s^2 \right)^{\frac{1}{2}} \left(\sum_{j \in \mathbb{Z}^d, k \in V_m} \lambda_m^j d_{\mathfrak{C}}(\omega^{j,k}, \beta^{j,k})^2 \right)^{\frac{1}{2}} \\
& + \sqrt{\rho} |V_m|^{\frac{1}{2}} \left(\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|Z^j\|_s^2 \right)^{\frac{1}{2}} \left(\sum_{j \in \mathbb{Z}^d, k \in V_m} \lambda_m^j d_{\mathfrak{C}}(\omega^{j,k}, \beta^{j,k})^2 \right)^{\frac{1}{2}} \\
& + \sum_{j \in \mathbb{Z}^d, k \in V_n - V_m} \lambda_m^j |\beta^{j,k}|_{\mathfrak{C}} (\|Z^j\|_s + 1) \Big] ds,
\end{aligned}$$

by (54), and using Jensen's Inequality to obtain the bound

$\left(\sum_{k \in V_m} d_{\mathfrak{C}}(\omega^{j,k}, \beta^{j,k}) \right)^2 \leq |V_m| \sum_{k \in V_m} d_{\mathfrak{C}}(\omega^{j,k}, \beta^{j,k})^2$. Hence by Gronwall's Inequality,

$$\begin{aligned}
& \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|X^j - Z^j\|_T \leq \exp(TC + T(1 + \rho)C_J |V_m|) \\
& \times \left[(1 + \sqrt{\rho})T |V_m|^{\frac{1}{2}} \left(\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|Z^j\|_T^2 \right)^{\frac{1}{2}} \left(\sum_{j \in \mathbb{Z}^d, k \in V_m} \lambda_m^j d_{\mathfrak{C}}(\omega^{j,k}, \beta^{j,k})^2 \right)^{\frac{1}{2}} \right. \\
& \left. + 2 \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|Q^j - R^j\|_T + T \sum_{j \in \mathbb{Z}^d, k \in V_n - V_m} \lambda_m^j |\beta^{j,k}|_{\mathfrak{C}} (\|Z^j\|_T + 1) \right].
\end{aligned}$$

Lemma 7 Suppose that $\mathcal{Z} = (R^j, \omega^{j,k})_{j,k \in \mathbb{Z}^d} \in \bar{\mathcal{T}}^{\mathbb{Z}^d}$ and $\Psi^m(\mathcal{Z})$ exists. Then

$$\begin{aligned} \|\Psi^m(\mathcal{Z})^j\|_T &\leq \exp\left(T\left(C + 2 \sum_{k \in V_m} |\omega^{j,k}|_{\mathfrak{C}}\right)\right) \\ &\quad + 2 \exp\left(T\left(C + \sum_{k \in V_m} |\omega^{j,k}|_{\mathfrak{C}}\right)\right) \|R^j\|_T. \end{aligned}$$

Proof Suppose that $[\tau, \gamma] \subseteq [0, T]$ is such that $\Psi^m(\mathcal{Z})_t^j \geq 0$ for all $t \in [\tau, \gamma]$. Then, making use of Assumption 1, for $t \in [\tau, \gamma]$,

$$\begin{aligned} |\Psi^m(\mathcal{Z})_t^j| &\leq |\Psi^m(\mathcal{Z})_\tau^j| + R_\tau^j - R_\tau^j \\ &\quad + \int_\tau^t \left(C \|\Psi^m(\mathcal{Z})^j\|_s + \sum_{k \in V_m} |\omega^{j,k}|_{\mathfrak{C}} (1 + \|\Psi^m(\mathcal{Z})^j\|_s) \right) ds. \end{aligned}$$

We take τ to be such that, either $\tau = 0$ (in which case $\Psi^m(\mathcal{Z})_0^j = R_0^j$) or $\Psi^m(\mathcal{Z})_\tau^j = 0$. Whichever is the case, we find that

$$|\Psi^m(\mathcal{Z})_t^j| \leq \int_0^t \left(C \|\Psi^m(\mathcal{Z})^j\|_s + \sum_{k \in V_m} |\omega^{j,k}|_{\mathfrak{C}} (1 + \|\Psi^m(\mathcal{Z})^j\|_s) \right) ds + 2 \|R^j\|_t.$$

We would obtain the same inequality if we had assumed that $\Psi^m(\mathcal{Z})_t^j \leq 0$ for all $t \in [\tau, \gamma]$. Since any $t \in [0, T]$ must satisfy either $\Psi^m(\mathcal{Z})_t^j \geq 0$ or $\Psi^m(\mathcal{Z})_t^j \leq 0$, we find that for all $t \in [0, T]$,

$$\|\Psi^m(\mathcal{Z})^j\|_t \leq \int_0^t \left(C \|\Psi^m(\mathcal{Z})^j\|_s + \sum_{k \in V_m} |\omega^{j,k}|_{\mathfrak{C}} (1 + \|\Psi^m(\mathcal{Z})^j\|_s) \right) ds + 2 \|R^j\|_t.$$

By Gronwall's Inequality,

$$\|\Psi^m(\mathcal{Z})^j\|_T \leq \exp\left(T\left(C + \sum_{k \in V_m} |\omega^{j,k}|_{\mathfrak{C}}\right)\right) \left(2 \|R^j\|_T + T \sum_{k \in V_m} |\omega^{j,k}|_{\mathfrak{C}}\right).$$

The lemma follows directly from this.

The following lemma is needed to prove Lemma 10.

Lemma 8 For any $j \in \mathbb{Z}^d$,

$$\liminf_{m \rightarrow \infty} |V_m| \lambda_m^j \geq \frac{\rho - 1}{\rho^2}.$$

Proof We first bound h using (50). Observe that, since $\tilde{\kappa}_m(\theta) \leq |V_m|$,

$$\frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \left(\rho |V_m| - \tilde{\kappa}_m(\theta) \right)^{-1} d\theta \leq \frac{1}{(2\pi)^d (\rho - 1)} \int_{[-\pi, \pi]^d} |V_m|^{-1} d\theta = \frac{1}{(\rho - 1) |V_m|}.$$

This means that $h \geq |V_m|(\rho - 1)$. We use a Taylor expansion to bound λ_m^j as follows,

$$\begin{aligned} \lambda_m^j &= \frac{h}{(2\pi)^d} \int_{[-\pi, \pi]^d} \exp(i\langle j, \theta \rangle) \left(\rho |V_m| - \tilde{\kappa}_m(\theta) \right)^{-1} d\theta \\ &= \frac{h}{\rho |V_m| (2\pi)^d} \sum_{k=0}^{\infty} \int_{[-\pi, \pi]^d} \exp(i\langle j, \theta \rangle) \left(\frac{\tilde{\kappa}_m(\theta)}{\rho |V_m|} \right)^k d\theta \\ &\geq \frac{h}{\rho |V_m| (2\pi)^d} \int_{[-\pi, \pi]^d} \exp(i\langle j, \theta \rangle) \left(\frac{\tilde{\kappa}_m(\theta)}{\rho |V_m|} \right) d\theta. \end{aligned} \quad (79)$$

This last step is due to the fact that for any $k \in \mathbb{Z}^+$

$$\int_{[-\pi, \pi]^d} \exp(i\langle j, \theta \rangle) \left(\frac{\tilde{\kappa}_m(\theta)}{\rho |V_m|} \right)^k d\theta \geq 0.$$

For example, if $k = 2$ in the above, then from the convolution formula for Fourier Series,

$$\frac{1}{\rho |V_m| (2\pi)^d} \int_{[-\pi, \pi]^d} \exp(i\langle j, \theta \rangle) \left(\frac{\tilde{\kappa}_m(\theta)}{\rho |V_m|} \right)^k d\theta = \frac{1}{\rho^3 |V_m|^3} \sum_{r \in \mathbb{Z}^d} \kappa_m^{j-r} \kappa_m^r.$$

This is non-negative because each κ_m^r is non-negative. This result easily generalises to $k > 2$. Coming back to (79), we see that

$$\lambda_m^j \geq \frac{h}{\rho |V_m| (2\pi)^d} (2\pi)^d \frac{\kappa_m^j}{\rho |V_m|} \geq \frac{|V_m|(\rho - 1)}{\rho^2 |V_m|^2} \kappa_m^j, \quad (80)$$

since $h \geq |V_m|(\rho - 1)$. Once m is large enough that $j \in V_m$, we have the lemma.

6 Lemmas Auxiliary to the Proof of Theorem 1

The three main results of this Section are Lemmas 9, 10 and 11. They are all needed to complete the proof of Theorem 1. For the following lemma recall that \mathcal{A}_c is defined in (39).

Lemma 9

$$\lim_{c \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} \frac{1}{|V_n|} \log \mathbb{P} \left(\hat{\mu}^n(\mathcal{Y}^n) \notin \mathcal{A}_c \right) = -\infty. \quad (81)$$

Proof For any $m > \mathfrak{m}_0$,

$$\begin{aligned} \mathbb{P}\left(\hat{\mu}^n(\mathcal{Y}^n) \notin \mathcal{A}_c\right) &\leq \mathbb{P}\left(\sum_{j \in V_n} \|W^{n,j}\|_T^2 > c|V_n|\right) + \\ &\mathbb{P}\left(|V_m|^{2+2\rho} \exp[(4+2\rho)TC_J|V_m|] \sum_{j \in V_n} \left(\sum_{k \notin V_m} |J^{n,j,k}|_{\mathfrak{C}}\right)^2 > c|V_n|\right) \\ &+ \mathbb{P}\left(|V_m|^{1+\rho} \exp[(3+\rho)TC_J|V_m|] \sum_{j \in V_n} \sum_{k \notin V_m} |J^{n,j,k}|_{\mathfrak{C}} > c|V_n|\right). \end{aligned}$$

We bound these two terms using the exponential Chebyshev Inequality. Using Assumption 5,

$$\begin{aligned} \mathbb{P}\left(\sum_{j \in V_n} \|W^{n,j}\|_T^2 > c|V_n|\right) &= \mathbb{P}\left(\mathfrak{c}_1 \sum_{j \in V_n} \|W^{n,j}\|_T^2 > \mathfrak{c}_1 c|V_n|\right) \\ &\leq \exp(-\mathfrak{c}_1 c|V_n|) \mathbb{E}\left[\exp\left(\mathfrak{c}_1 \sum_{j \in V_n} \|W^{n,j}\|_T^2\right)\right] \\ &\leq \exp(-\mathfrak{c}_1 c|V_n| + \mathfrak{c}_2 |V_n|). \end{aligned}$$

Making similar use of Assumption 3, we find that

$$\begin{aligned} \mathbb{P}\left(|V_m|^{2+2\rho} \exp[(4+2\rho)TC_J|V_m|] \sum_{j \in V_n} \left(\sum_{k \in V_m} |J^{n,j,k}|_{\mathfrak{C}}\right)^2 > c|V_n|\right) \\ \leq \exp(-c\mathfrak{a}_1|V_n| + \mathfrak{a}_2|V_n|) \\ \mathbb{P}\left(|V_m|^{1+\rho} \exp[(3+\rho)TC_J|V_m|] \sum_{j \in V_n, k \in V_m} |J^{n,j,k}|_{\mathfrak{C}} > c|V_n|\right) \\ \leq \exp(-c\mathfrak{a}_1|V_n| + \mathfrak{a}_2|V_n|). \end{aligned}$$

We thus see that

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{|V_n|} \log \mathbb{P}\left(\hat{\mu}^n(\mathcal{Y}^n) \notin \mathcal{A}_c\right) \leq \max\{-c\mathfrak{a}_1 + \mathfrak{a}_2, -\mathfrak{c}_1 c + \mathfrak{c}_2\}.$$

The lemma follows directly from the above.

Lemma 10 *For any $c, \delta > 0$, there exists an $m \in \mathbb{Z}^+$ such that*

$$\sup_{n \geq m} \sup_{\mu \in \mathcal{A}_c} d^{\mathcal{P}}(\tilde{\Psi}^m(\mu), \tilde{\Psi}^n(\mu)) \leq \delta. \quad (82)$$

For all $\mu \in \cup_{c \geq 0} \mathcal{A}_c$,

$$\mu \circ \Psi^{-1} = \lim_{m \rightarrow \infty} \tilde{\Psi}^m(\mu). \quad (83)$$

Proof Let $r \in \mathbb{Z}^+$ be such that $2^{-r} \leq \frac{\delta}{2}$. It follows from the definition of the metric $d^{\mathcal{P}}$ in (2) that

$$d^{\mathcal{P}}(\tilde{\Psi}^m(\mu), \tilde{\Psi}^n(\mu)) \leq 2^{-r} + d_r^{\mathcal{P}}(\tilde{\Psi}^m(\mu), \tilde{\Psi}^n(\mu)).$$

For (82), it thus suffices for us to show that for fixed $r \in \mathbb{Z}^+$ and $\delta > 0$, there exists an $m \in \mathbb{Z}^+$ such that

$$\sup_{n \geq m} \sup_{\mu \in \mathcal{A}_c} d_r^{\mathcal{P}}(\tilde{\Psi}^m(\mu), \tilde{\Psi}^n(\mu)) \leq \delta. \quad (84)$$

Let $\mathcal{X}^j = (Q^j, (\omega^{j,k})_{k \in \mathbb{Z}^d})$, $X = \Psi^m(\mathcal{X})$ and $Z = \Psi^n(\mathcal{X})$. By Lemma 6,

$$\begin{aligned} \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|X^j - Z^j\|_T &\leq T \exp(T(2C + (1 + \rho)C_J|V_m|)) \times \\ &\quad \sum_{j \in \mathbb{Z}^d, k \in V_n - V_m} \lambda_m^j |\omega^{j,k}|_{\mathfrak{C}} (\|Z^j\|_T + 1). \end{aligned}$$

We multiply both sides of the above equation by $|V_m|$ and use Lemma 7, finding that

$$\begin{aligned} |V_m| \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|X^j - Z^j\|_T &\leq |V_m| T \exp(2TC) \times \\ &\quad \sum_{j \in \mathbb{Z}^d, k \in V_n - V_m} \lambda_m^j |\omega^{j,k}|_{\mathfrak{C}} \left[\exp((3 + \rho)TC_J|V_m|) \right. \\ &\quad \left. + \exp((2 + \rho)TC_J|V_m|) \|Q^j\|_T + \exp((1 + \rho)TC_J|V_m|) \right]. \end{aligned}$$

Let m be sufficiently large that

$$\sup_{j \in V_r} |V_m| \lambda_m^j \geq \frac{\rho - 1}{\rho^2},$$

which is possible thanks to Lemma 8. We then use Chebyshev's Inequality to find that

$$\begin{aligned}
& \mu\left(\sum_{j \in V_r} \|X^j - Z^j\|_T > \epsilon\right) \leq \mu\left(\frac{\rho^2}{\rho-1}|V_m| \sum_{j \in \mathbb{Z}^d} \lambda_m^j \|X^j - Z^j\|_T > \epsilon\right) \\
& \leq \frac{\rho^2}{\rho-1} \epsilon^{-1} |V_m| T \exp(2TC) \mathbb{E}^\mu \left[2 \exp((3+\rho)TC_J|V_m|) \sum_{j \in \mathbb{Z}^d, k \in V_n - V_m} \lambda_m^j |\omega^{j,k}|_{\mathfrak{C}} \right. \\
& \quad \left. + \exp((2+\rho)TC_J|V_m|) \sum_{j \in \mathbb{Z}^d, k \in V_n - V_m} \lambda_m^j |\omega^{j,k}|_{\mathfrak{C}} \|Q^j\|_T \right] \\
& \leq \frac{2\rho^2}{\rho-1} c \epsilon^{-1} T \exp(2TC) |V_m|^{-\rho} + \frac{\rho^2}{\rho-1} \epsilon^{-1} |V_m| T \exp(2TC + (2+\rho)TC_J|V_m|) \times \\
& \quad \mathbb{E}^\mu \left[\sum_{j \in \mathbb{Z}^d} \lambda_m^j \left(\sum_{k \in V_n - V_m} |\omega^{j,k}|_{\mathfrak{C}} \right)^2 \right]^{\frac{1}{2}} \times \mathbb{E}^\mu \left[\sum_{j \in \mathbb{Z}^d} \lambda_m^j \|Q^j\|_T^2 \right]^{\frac{1}{2}} \\
& \leq \frac{2\rho^2 c}{\rho-1} \epsilon^{-1} T \exp(2TC) |V_m|^{-\rho} + \frac{\rho^2}{\rho-1} c \epsilon^{-1} |V_m|^{-\rho} T \exp(2TC),
\end{aligned} \tag{85}$$

where we have used the Cauchy-Schwarz Inequality, and twice used the definition of \mathcal{A}_c in (39). Since $\rho > 1$, for m sufficiently large the above equation is less than δ , yielding (82). We now prove (83). Let $\{p_k\}_{k=1}^\infty \subset \mathbb{Z}^+$ be such that $p_k > k$ and

$$\mu\left(\sum_{j \in V_r} \|\Psi^{p_k}(\mathcal{X})^j - \Psi^{p_{k+1}}(\mathcal{X})^j\|_T > 2^{-k}\right) \leq 2^{-k}.$$

It follows from the Borel-Cantelli Lemma and the two previous equations that $\{\Psi^{p_k}(\mathcal{X})\}_{k=1}^\infty$ converges almost surely. Let $Y^j = \lim_{k \rightarrow \infty} \Psi^{p_k}(\mathcal{X})^j$ and $\epsilon_m = |V_m|^{\frac{1}{2}(1-\rho)}$. Then it follows from (85) that, since $\rho > 1$,

$$\sum_{m=1}^\infty \mu\left(\sum_{j \in V_r} \|\Psi^m(\mathcal{X}) - \Psi^{p_m}(\mathcal{X})\| > \epsilon_m\right) \leq \frac{3\rho^2 c}{\rho-1} T \exp(2TC) \sum_{m=1}^\infty |V_m|^{-\frac{1+\rho}{2}} < \infty. \tag{86}$$

By the Borel-Cantelli Lemma, for all $j \in V_r$, $\Psi^m(\mathcal{X})^j \rightarrow Y^j$ almost surely. It remains for us to show that Y satisfies the relation in (40). It suffices to show that for all $t \in [0, T]$,

$$\int_0^t \sum_{k \in \mathbb{Z}^d} \Lambda_s^k(\omega^{j,k}, \Psi^m(\mathcal{X})^j, \Psi^m(\mathcal{X})^{j+k}) ds \rightarrow \int_0^t \sum_{k \in \mathbb{Z}^d} \Lambda_s^k(\omega^{j,k}, Y^j, Y^{j+k}) ds \tag{87}$$

$$\int_0^t \mathfrak{b}_s(\Psi^m(\mathcal{X})^j) ds \rightarrow \int_0^t \mathfrak{b}_s(Y^j) ds \tag{88}$$

It follows from an application of the Borel-Cantelli Lemma to (39) that, μ almost surely, there must exist some $q \in \mathbb{Z}^+$ such that $\omega^{j,k} = \mathfrak{N}$ for all $k \notin V_q$.

Since r is arbitrary, we may assume that $r \geq q$. Since $\Psi^m(\mathcal{X})^{j+k} \rightarrow Y^{j+k}$ for all $k \in V_q$, (87) follows from Assumption 2 and the dominated convergence theorem. (88) follows from Assumption 1 and the dominated convergence theorem.

In the following, recall the definition of \mathcal{A}_c in (39).

Lemma 11 *For each $\alpha \in \mathbb{R}^+$, there exists $c \in \mathbb{R}^+$ such that*

$$\left\{ \mu \in \mathcal{P}_{\bar{S}}(\bar{\mathcal{T}}^{\mathbb{Z}^d}) \mid I_{\mathcal{Y}}(\mu) \leq \alpha \right\} \subseteq \mathcal{A}_c. \quad (89)$$

Proof Suppose that $I_{\mathcal{Y}}(\mu) \leq \alpha$. By [22, Theorem 4.5.10], for all $f \in C_b(\bar{\mathcal{T}}^{\mathbb{Z}^d})$, where $C_b(\bar{\mathcal{T}}^{\mathbb{Z}^d})$ is the set of all bounded continuous functions on $\bar{\mathcal{T}}^{\mathbb{Z}^d}$ which are $\mathcal{B}(\mathcal{T}^{\bar{V}_q})$ -measurable for some $q \in \mathbb{Z}^+$,

$$\mathbb{E}^\mu[f] - \overline{\lim}_{n \rightarrow \infty} \frac{1}{|V_n|} \log \mathbb{E} \left[\exp \left(\sum_{j \in V_n} f(\bar{S}^j \tilde{\mathcal{Y}}^n) \right) \right] \leq \alpha. \quad (90)$$

We will show that in order that (89) is satisfied, it suffices to take

$$c = \max \left\{ \frac{\alpha + \mathfrak{c}_2}{\mathfrak{c}_1}, \frac{\alpha + \mathfrak{a}_2}{\mathfrak{a}_1} \right\}, \quad (91)$$

where the constants $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}_1, \mathfrak{a}_2$ are defined in Assumptions 3 and 5. For $r \in \mathbb{Z}^+$, let $\phi_r(\mathcal{Y}^n) := \mathfrak{c}_1 \sigma_r(\|W^{n,0}\|_T^2)$, where $\sigma_r : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous and bounded, and $\sigma_r(x) \uparrow x$ as $r \rightarrow \infty$. Substituting $f = \phi_r$ into (90), taking $r \rightarrow \infty$, and using Assumption 5, we find that

$$\mathfrak{c}_1 \mathbb{E}^\mu \left[\|X^0\|_T^2 \right] \leq \alpha + \mathfrak{c}_2. \quad (92)$$

This means that if $c \geq (\alpha + \mathfrak{c}_2)/\mathfrak{c}_1$, then the first condition for membership of \mathcal{A}_c is satisfied (see (39)). Next, we define for $r \in \mathbb{Z}^+$, $r > m \geq \mathfrak{m}_0$

$$\phi_r(\mathcal{Y}^n) = \mathfrak{a}_1 |V_m|^{2\rho+2} \exp \left(2T(2+\rho)C_J |V_m| \right) \left(\sum_{k \in V_r - V_m} |J^{n,0,k}|_{\mathfrak{e}} \right)^2.$$

Substituting $f = \phi_r$ into (90), taking $r \rightarrow \infty$, and using Assumption 3 we find that

$$\mathbb{E}^\mu \left[\mathfrak{a}_1 |V_m|^{2\rho+2} \exp \left(2T(2+\rho)C_J |V_m| \right) \sum_{j \in V_n} \left(\sum_{k \notin V_m} |J^{n,j,k}|_{\mathfrak{e}} \right)^2 \right] \leq \alpha + \mathfrak{a}_2.$$

This means that the second condition for membership of \mathcal{A}_c is satisfied once $c \geq \mathfrak{a}_1^{-1}(\alpha + \mathfrak{a}_2)$. We similarly find using Assumption 3 that

$$\mathbb{E}^\mu \left[\mathfrak{a}_1 |V_m|^{\rho+1} \exp \left(T(3+\rho)C_J |V_m| \right) \sum_{j \in V_n} \sum_{k \notin V_m} |J^{n,j,k}|_{\mathfrak{e}} \right] \leq \alpha + \mathfrak{a}_2,$$

which means that the third condition for membership of \mathcal{A}_c is satisfied once $c \geq \mathfrak{a}_1^{-1}(\alpha + \mathfrak{a}_2)$.

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References

1. G. ANDERSON, A. GUIONNET, AND O. ZEITOUNI, *An introduction to random matrices*, no. 1957, Springer Science & Business Media, 2009.
2. F. M. ATAY, S. BANISCH, P. BLANCHARD, B. CESSAC, E. OLBRICH, AND D. VOLCHENKOV, *Perspectives on multi-level dynamics*, arXiv preprint arXiv:1606.05665, (2016).
3. J. BALADRON, D. FASOLI, O. FAUGERAS, AND J. TOUBOUL, *Mean-field description and propagation of chaos in networks of Hodgkin-Huxley and Fitzhugh-Nagumo neurons*, The Journal of Mathematical Neuroscience, 2 (2012).
4. A. BARRA AND E. AGLIARI, *Equilibrium statistical mechanics on correlated random graphs*, Journal of Statistical Mechanics: Theory and Experiment, 2011 (2011), p. P02027.
5. G. BEN-AROUS AND A. GUIONNET, *Large deviations for langevin spin glass dynamics*, Probability Theory and Related Fields, 102 (1995), pp. 455–509.
6. P. BILLINGSLEY, *Convergence of Probability Measures*, Wiley series in probability and statistics, 1999.
7. M. BOSSY, O. FAUGERAS, AND D. TALAY, *Clarification and complement to “mean-field description and propagation of chaos in networks of hodgkin-huxley and fitzhugh-nagumo neurons”*, The Journal of Mathematical Neuroscience (JMN), 5 (2015), pp. 1–23.
8. P. BRESSLOFF, *Stochastic neural field theory and the system-size expansion*, SIAM J. Appl. Math, 70 (2009), pp. 1488–1521.
9. P. BRESSLOFF, *Spatiotemporal dynamics of continuum neural fields*, Journal of Physics A: Mathematical and Theoretical, 45 (2012).
10. A. BUDHIRAJA, P. DUPUIS, AND M. FISCHER, *Large deviation properties of weakly interacting processes via weak convergence methods*, Annals of Probability, 40 (2012), pp. 74–102.
11. M. BUICE, J. COWAN, AND C. CHOW, *Systematic fluctuation expansion for neural network activity equations*, Neural computation, 22 (2010), pp. 377–426.
12. T. CABANA AND J. TOUBOUL, *Large deviations, dynamics and phase transitions in large stochastic and disordered neural networks*, Journal of Statistical Physics, 153 (2013), pp. 211–269.
13. —, *Large deviations of particle systems in random interaction*, arXiv preprint arXiv:1601.00985, (2016).
14. B. CESSAC AND R. COFRE, *Spike train statistics and gibbs distributions*, Journal of Physiology-Paris, 107 (2013), pp. 360–368.

15. S. CHATTERJEE AND S. VARADHAN, *The large deviation principle for the erdős-rényi random graph*, European Journal of Combinatorics, 32 (2011), pp. 1000–1017.
16. F. COMETS AND A. DEMBO, *Ordered overlaps in disordered mean-field models*, Probability theory and related fields, 121 (2001), pp. 1–29.
17. J. COX, K. FLEISCHMANN, AND A. GREVEN, *Comparison of interacting diffusions and an application to their ergodic theory*, Probability Theory and Related Fields, 105 (1996), pp. 513–528.
18. P. DAI PRA AND F. DEN HOLLANDER, *McKean-vlasov limit for interacting random processes in random media*, Journal of statistical physics, 84 (1996), pp. 735–772.
19. D. DAWSON AND P. DEL MORAL, *Large deviations for interacting processes in the strong topology*, in Statistical Modeling and Analysis for Complex Data Problems, Springer US, 2005.
20. L. C. G. DEL MOLINO, K. PAKDAMAN, J. TOUBOUL, AND G. WAINRIB, *Synchronization in random balanced networks*, Physical Review E, 88 (2013), p. 042824.
21. F. DELARUE, J. INGLIS, S. RUBENTHALER, AND E. TANRÉ, *Particle systems with a singular mean-field self-excitation. application to neuronal networks*, Stochastic Processes and their Applications, 125 (2015), pp. 2451–2492.
22. A. DEMBO AND O. ZEITOUNI, *Large deviations techniques*, Springer, 1997. 2nd Edition.
23. A. DESTEXHE, Z. MAINEN, AND T. SEJNOWSKI, *An efficient method for computing synaptic conductances based on a kinetic model of receptor binding*, Neural Computation, 6 (1994), pp. 14–18.
24. S. DOMMERS, C. GIARDINÀ, AND R. VAN DER HOFSTAD, *Ising models on power-law random graphs*, Journal of Statistical Physics, 141 (2010), pp. 638–660.
25. M. DONSKER AND S. VARADHAN, *Large deviations for stationary Gaussian processes*, Commun. Math. Phys., 97 (1985), pp. 187–210.
26. J. DUGUNDJI, *An extension of tietze’s theorem*, Pacific J. Math, 1 (1951), pp. 353–367.
27. R. DURRETT, *Random graph dynamics*, vol. 200, Cambridge university press Cambridge, 2007.
28. R. ELLIS, *Entropy, large deviations and statistical mechanics*, Springer, 1985.
29. G. B. ERMENTROUT AND D. TERMAN, *Foundations of Mathematical Neuroscience*, Interdisciplinary Applied Mathematics, Springer, 2010.
30. D. FASOLI, A. CATTANI, AND S. PANZERI, *The complexity of dynamics in small neural circuits*, arXiv preprint arXiv:1506.08995, (2015).
31. O. FAUGERAS AND J. MACLAURIN, *A large deviation principle and an expression of the rate function for a discrete stationary gaussian process*, Entropy, 16 (2014), p. 6722.
32. O. FAUGERAS AND J. MACLAURIN, *Large deviations of a spatially-stationary network of interacting neurons*, arxiv depot, INRIA Sophia Antipolis, <http://arxiv.org/abs/1404.0732>, 2014.

-
33. O. FAUGERAS AND J. MACLAURIN, *Asymptotic description of neural networks with correlated synaptic weights*, Entropy, 17 (2015), p. 4701.
 34. M. FISCHER ET AL., *On the form of the large deviation rate function for the empirical measures of weakly interacting systems*, Bernoulli, 20 (2014), pp. 1765–1801.
 35. R. FITZHUGH, *Mathematical models of threshold phenomena in the nerve membrane*, Bulletin of Mathematical Biology, 17 (1955), pp. 257–278.
 36. R. FITZHUGH, *Theoretical Effect of Temperature on Threshold in the Hodgkin-Huxley Nerve Model*, The Journal of General Physiology, 49 (1966), pp. 989–1005.
 37. R. FITZHUGH, *Mathematical models of excitation and propagation in nerve*, McGraw-Hill Companies, 1969, ch. 1.
 38. R. FUKUSHIMA AND N. KUBOTA, *Quenched large deviations for multidimensional random walk in random environment with holding times*, Journal of Theoretical Probability, 27 (2014), pp. 1140–1166.
 39. H.-O. GEORGII, *Gibbs Measures and Phase Transitions*, De Gruyter, 1988.
 40. H.-O. GEORGII, *Large deviations and maximum entropy principle for interacting random fields on \mathbb{Z}^d* , The Annals of Probability, (1993), pp. 1845–1875.
 41. W. GERSTNER AND W. KISTLER, *Spiking Neuron Models*, Cambridge University Press, 2002.
 42. G. GIACOMIN, E. LUÇON, AND C. POQUET, *Coherence stability and effect of random natural frequencies in populations of coupled oscillators*, Journal of Dynamics and Differential Equations, 26 (2014), pp. 333–367.
 43. B. GOLDYS AND M. MUSIELA, *Infinite dimensional diffusions, kolmogorov equations and interest rate models*, Option Pricing, Interest Rates and Risk Management, (2001), pp. 314–335.
 44. A. GREVEN AND F. D. HOLLANDER, *Phase transitions for the long-time behavior of interacting diffusions*, The Annals of Probability, 35 (2007).
 45. A. GUIONNET AND B. ZEGARLINSKI, *Decay to equilibrium in random spin systems on a lattice*, Communications in mathematical physics, 181 (1996), pp. 703–732.
 46. N. KUBOTA ET AL., *Large deviations for simple random walk on supercritical percolation clusters*, Kodai Mathematical Journal, 35 (2012), pp. 560–575.
 47. T. M. LIGGETT, *Interacting Particle Systems*, Springer Berlin Heidelberg, 2005.
 48. M. LÖWE AND F. VERMET, *The hopfield model on a sparse erdős-renyi graph*, Journal of Statistical Physics, 143 (2011), pp. 205–214.
 49. E. LUÇON AND C. POQUET, *Long time dynamics and disorder-induced traveling waves in the stochastic kuramoto model*, arXiv preprint arXiv:1505.00497, (2015).
 50. E. LUÇON AND W. STANNAT, *Mean field limit for disordered diffusions with singular interactions*, The Annals of Applied Probability, 24 (2014), pp. 1946–1993.
 51. D. LUSHER, J. KOSKINEN, AND G. ROBINS, *Exponential random graph models for social networks: Theory, methods, and applications*, Cam-

-
- bridge University Press, 2012.
52. O. MOYNOT AND M. SAMUELIDES, *Large deviations and mean-field theory for asymmetric random recurrent neural networks*, Probability Theory and Related Fields, 123 (2002), pp. 41–75.
 53. J. NAGUMO, S. ARIMOTO, AND S. YOSHIKAWA, *An active pulse transmission line simulating nerve axon*, Proceedings of the IRE, 50 (1962), pp. 2061–2070.
 54. E. NIEBUR, H. G. SCHUSTER, D. M. KAMMEN, AND C. KOCH, *Oscillator-phase coupling for different two-dimensional network connectivities*, Physical Review A, 44 (1991), p. 6895.
 55. C.-E. PFISTER, *Thermodynamical aspects of classical lattice systems*, Progress in Probability, 51 (2002), pp. 393–472.
 56. C. QUIÑINAO AND J. TOUBOUL, *Limits and dynamics of randomly connected neuronal networks*, Acta Applicandae Mathematicae, 136 (2015), pp. 167–192.
 57. F. RASSOUL-AGHA AND T. SEPPÄLÄINEN, *Process-level quenched large deviations for random walk in random environment*, in Annales de l’institut Henri Poincaré (B), vol. 47, 2011, pp. 214–242.
 58. G. ROBINS, P. PATTISON, Y. KALISH, AND D. LUSHER, *An introduction to exponential random graph (p^*) models for social networks*, Social networks, 29 (2007), pp. 173–191.
 59. M. RUBINOV, O. SPORNS, J.-P. THIVIERGE, AND M. BREAKSPEAR, *Neurobiologically realistic determinants of self-organized criticality in networks of spiking neurons*, PLoS Comput Biol, 7 (2011), p. e1002038.
 60. T. SEPPÄLÄINEN, *Entropy, limit theorems, and variational principles for disordered lattice systems*, Communications in mathematical physics, 171 (1995), pp. 233–277.
 61. T. SHIGA AND A. SHIMIZU, *Infinite dimensional stochastic differential equations and their applications*, Journal Mathematics Kyoto University, 20 (1980), pp. 395–416.
 62. S. L. SIMPSON, S. HAYASAKA, AND P. J. LAURIENTI, *Exponential random graph modeling for complex brain networks*, PloS one, 6 (2011), p. e20039.
 63. H. SOMPOLINSKY, A. CRISANTI, AND H. SOMMERS, *Chaos in Random Neural Networks*, Physical Review Letters, 61 (1988), pp. 259–262.
 64. O. SPORNS, *Networks of the Brain*, MIT press, 2011.
 65. A.-S. SZNITMAN, *Topics in propagation of chaos*, in Ecole d’Eté de Probabilités de Saint-Flour XIX — 1989, D. Burkholder, E. Pardoux, and A.-S. Sznitman, eds., vol. 1464 of Lecture Notes in Mathematics, Springer Berlin / Heidelberg, 1991, pp. 165–251. 10.1007/BFb0085169.
 66. J. TOUBOUL, *Limits and dynamics of stochastic neuronal networks with random heterogeneous delays*, Journal of Statistical Physics, 149 (2012), pp. 569–597.
 67. J. TOUBOUL, *The propagation of chaos in neural fields*, The Annals of Applied Probability, 24 (2014).
 68. J. TOUBOUL AND B. ERMENTROUT, *Finite-size and correlation-induced effects in mean-field dynamics*, J Comput Neurosci, 31 (2011), pp. 453–484.

- 69. G. WAINRIB AND M. GALTIER, *Regular graphs maximize the variability of random neural networks*, Physical Review E, 92 (2015), p. 032802.
- 70. D. J. WATTS AND S. H. STROGATZ, *Collective dynamics of small-world networks*, Nature, 393 (1998).
- 71. B. ZEGARLINSKI, *Interactions and pressure functionals for disordered lattice systems*, Communications in mathematical physics, 139 (1991), pp. 305–339.
- 72. O. ZEITOUNI, *Part ii: Random walks in random environment*, in Lectures on probability theory and statistics, Springer, 1837, pp. 189–312.